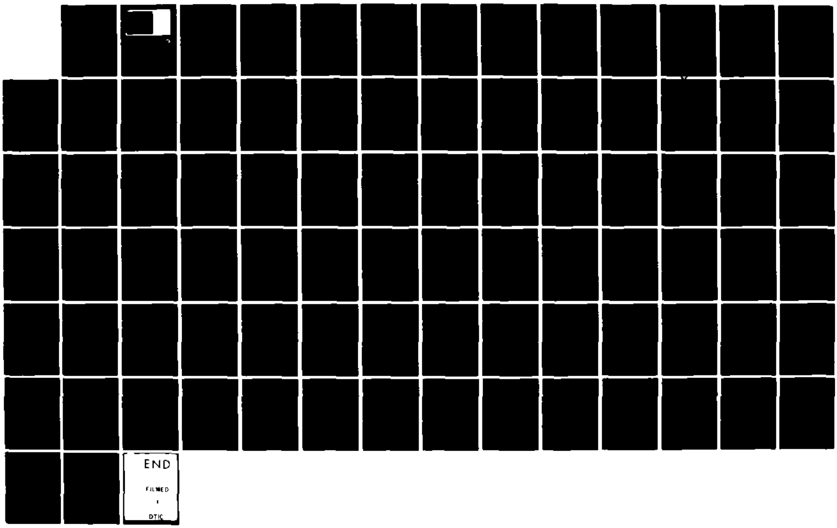
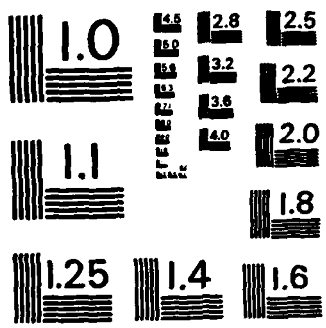
AD-A120 960 FINITE-AMPLITUDE STEADY WAVES IN STRATIFIED FLUIDS(U) 1/1
WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER
J L BONA ET AL. JUL 82 MRC-TSR-2401 DAAG29-80-C-0041

UNCLASSIFIED F/G 12/1 NL





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

MRC Technical Summary Report #2401

FINITE-AMPLITUDE STEADY WAVES IN STRATIFIED FLUIDS

J. L. Bona, D. K. Bose, R. E. L. Turner

Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

July 1982

(Received September 2, 1981)

Sponsored by

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

National Science Foundation
Washington, DC 20550

UNIVERSITY OF WISCONSIN - MADISON
MATHEMATICS RESEARCH CENTER

FINITE-AMPLITUDE STEADY WAVES IN STRATIFIED FLUIDS

J. L. Bona[†,1,2], D. K. Bose[††,3] and R. E. L. Turner[*,1,4]

Technical Summary Report #2401

July 1982

ABSTRACT

An exact theory regarding solitary internal gravity waves in stratified fluids is presented. Two-dimensional, inviscid, incompressible flows confined between plane horizontal rigid boundaries are considered. Variational techniques are used to demonstrate that the Euler equations possess solutions that represent progressing waves of permanent form. These are analogous to the surface, solitary waves so easily generated in a flume. Periodic wave trains of permanent form, the analogue of the classical cnoidal waves, are also found. Moreover, internal solitary-wave solutions are shown to arise as the limit of cnoidal wave trains as the period length grows unboundedly.

AMS (MOS) Subject Classifications: 35J20, 35J60, 76B25, 76C10

Key Words: Internal wave, solitary wave, cnoidal wave, critical point, symmetrization, bifurcation

Work Unit Number 1 - Applied Analysis

[†] Department of Mathematics, University of Chicago and Mathematics Research Center, University of Wisconsin-Madison, U.S.A.

[††] Department of Mathematics, Brighton Polytechnic, U.K.

[*] Department of Mathematics and Mathematics Research Center, University of Wisconsin-Madison, U.S.A.

---

[1] Sponsored by the United States Army under Contract No. DAAG29-80-C-0041;

[2] the National Science Foundation (U.S.A.) under Grant No. MCS-8002327;

[3] the Science Research Council, U.K.;

[4] the National Science Foundation (U.S.A.) under Grant No. MCS-7904426.

SIGNIFICANCE AND EXPLANATION

The notion of a solitary wave arose in the last century and provoked some controversy as to the existence of such a phenomenon. After more than half a century of only sporadic interest in these waves, the last twenty years have seen an upsurgence of scientific work on the subject. The interest stems from the realization that these special wave forms play a significant role in the evolution of general classes of disturbances. This property of solitary waves is observed on a particularly grand scale in the earth's oceans. There, disturbances that appear to be internal solitary waves with crests spanning hundreds of kilometers have been recorded. Various approximate models have been used to analyze these motions. Nevertheless, we are far from a complete mathematical treatment of the phenomena just described.

In this report a model physical problem is studied in a mathematically exact formulation. We consider two-dimensional flows confined between rigid horizontal boundaries and show that there are solutions of the Euler equations representing internal solitary gravity waves. Our theory, which is not restricted to small amplitudes predicts both waves of elevation and depression, depending on the ambient density distribution and the velocity distribution at infinity. Just as for the classical surface solitary waves, these waves are single-crested, symmetric, and decay exponentially away from the crest. They thus represent disturbances of essentially finite extent. These qualitative features are established using variational techniques combined with the theory of rearrangements. Moreover, it is shown that the single-crested wave arises as a limit of periodic wave trains of increasing wave lengths.

FINITE-AMPLITUDE STEADY WAVES IN STRATIFIED FLUIDS

J. L. Bona[†,1,2], D. K. Bose[††,3] and R. E. L. Turner[*,1,4]

1. INTRODUCTION

The mathematical problem to be analysed in this paper arises in the study of steady, two-dimensional wave motion in heterogeneous, inviscid, incompressible fluids confined between two rigid horizontal planes. The existence and various properties of finite-amplitude periodic wave trains, and single-crested waves, of permanent form will be established. These internal waves are analogous, respectively, to cnoidal and solitary surface waves, and will be given these names in what follows.

The first rigorous existence theory for internal waves appears to have been presented by Kotschin (1928). He considered a system comprised of two homogeneous layers of fluid of differing density, the lighter resting on the heavier, and the whole contained between two fixed horizontal plane boundaries. Kotschin used a majorant method to obtain small-amplitude cnoidal-wave solutions of the Euler equations posed by this two-fluid system. The first rigorous theory to include both cnoidal and solitary

† Department of Mathematics, University of Chicago and the Mathematics Research Center, University of Wisconsin-Madison, U.S.A.

†† Department of Mathematics, Brighton Polytechnic, U.K.

* Department of Mathematics and the Mathematics Research Center, University of Wisconsin-Madison, U.S.A.

1 Sponsored by the United States Army under Contract No. DAAG29-80-C-0041;

2 the National Science Foundation (U.S.A.) under Grant No. MCS-8002327;

3 the Science Research Council, U.K.;

4 the National Science Foundation (U.S.A.) under Grant No. MCS-7904426.

internal waves was given by Ter-Krikorov (1963). He considered continuously stratified fluids with a rigid bottom boundary and either a fixed or a free top boundary. Ter-Krikorov relied on Yih's (1960) version of Long's (1953) equation and on the approximate description of the waves in question that are valid for small amplitudes and long wavelengths. Casting the equation satisfied by the difference between the exact and approximate solutions into operator form, he applied the contraction-mapping theorem to this operator, considered as a mapping of a small ball centered at the zero function in an appropriate function space. In this manner, he was able to deal with both cnoidal and solitary internal waves, but only of very small amplitude. His work thus compares with that of Lavrentief (1943, 1947), Friedrichs and Hyers (1954) and Littman (1957) concerning surface waves of permanent form in liquid of finite depth. Zeidler (1971), using complex function theory, reduced the steady internal-wave problem in the above-described two-fluid system to a situation in which he could demonstrate that a bifurcation at a simple eigenvalue occurs. The bifurcating branch thus gave small-amplitude cnoidal waves, the general result being closely allied in spirit to that of Kotschin. Going beyond Kotschin's theory, Zeidler was able to take account of capillarity in his theory. Benjamin (1973) gave an exact treatment of cnoidal internal waves whose amplitudes need not be small. He made use of positive-operator methods, as pioneered by Krasnosel'skii, together with the Leray-Schauder degree theory. His method did not yield a theory for solitary waves, though some mathematical evidence for their existence was presented.

The present treatment of the problem is very much influenced by Benjamin's (1973) wide-ranging article, though our methods are different. Benjamin's analysis has the speed of propagation as a fixed parameter whilst the energy possessed by the wave is left free. This problem is covered here,

as well as the complimentary specification in which the energy is fixed and the speed of propagation is left free. As explained in section 7, this latter case is probably more relevant to typical situations in which such waves are generated. Our variational methods, when combined with the use of rearrangement inequalities, are effective for a broad range of non-linearities. This combination of techniques has been fruitfully used in other hydrodynamic problems (see e.g. Friedrichs 1934; Garabedian 1965; Fraenkel and Berger 1973) and our treatment of constrained variational problems has several points in common with the last article cited. The use of symmetrization in the problem of fixed speed appears to be new.

In an outgrowth of the present paper, Turner (1982) has used an alternate variational principle to establish the existence of cnoidal and solitary waves in a fluid with a rapidly varying heterogeneity, the two-fluid system mentioned earlier being a special case of his theory.

The plan of the paper is as follows. Section 2 is devoted to a general description of the mathematical problem that presents itself in considering internal waves of permanent form. In section 3, existence of cnoidal internal waves is established, both for the case where the speed of propagation is given and for the situation in which the total energy is fixed. *A priori* bounds satisfied by these periodic solutions are deduced in section 4. These bounds are independent of the period length, and include a result of exponential decay from crest to trough. In section 5 internal solitary waves are shown to arise as the limiting forms of internal cnoidal waves as the period length becomes indefinitely large. Also discussed is a proposition implying the absence of closed streamlines. This technical point is important for interpreting the solutions obtained in our analysis as realizable wave motion. A broad class of concentration profiles is examined in section 6, and

shown to yield equations that fall within the confines of our theory. The relationship of our theory to other studies is considered in section 7. Field and laboratory evidence concerning internal waves of permanent form is briefly reviewed. Various implications and drawbacks of the theory developed herein are presented, along with suggestions of avenues for further investigation.

2. THE GOVERNING EQUATIONS

To understand certain aspects of the theory to be developed presently, the derivation and import of the equations that will hold our attention must be kept in mind. It is therefore useful to review some points arising in the formulation of the mathematical problem to be analysed. The description given here follows closely Benjamin's (1973) account, but the gist of it goes back much further (cf. Benjamin 1966, Dubreil-Jacotin 1935, Long 1963 and especially Yih 1958, 1960).

The idealized physical situation to be modeled is a heterogeneous incompressible fluid moving between horizontal plane boundaries. Interest will be focused on two-dimensional flows, which will be assumed to be inviscid and non-diffusive. These flows will be described in a two-dimensional Cartesian coordinate frame, in which the x-axis lies along the bottom flow boundary and height y above this boundary comprises the second coordinate. The distance between the boundaries is taken as the unit of length, so the lower and upper boundaries are represented by $\{(x,0) : x \in \mathbb{R}\}$ and $\{(x,1) : x \in \mathbb{R}\}$, respectively.

There is postulated a primary or base flow in which the fluid motion is everywhere horizontal with velocity $U(y)$ at height y above the lower boundary, $0 \leq y \leq 1$. It is also assumed that, in the primary state, the fluid density ρ is a decreasing function of y, for $0 \leq y \leq 1$. In fact, it will be assumed that ρ is a continuously differentiable function of y, and that $\rho'(y) < 0$, for y in $[0,1]$.

We search for waves of permanent form whose velocity of propagation downstream (in the direction of increasing x) is $\bar{c}$. It is convenient then to presume that the coordinate system is also moving downstream at speed $\bar{c}$, thus rendering stationary the waveform in question. In the moving frame of reference, the primary fluid velocity is $W(y) = U(y) - \bar{c}$.

Let $\underset{\sim}{q} = (u,v)$ denote the velocity field of a steady flow of interest. The flow is assumed to be incompressible and non-diffusive. The former assumption means $\underset{\sim}{q}$ is divergence free, whilst the latter amounts to the assertion that the derivative of the density following the flow is zero. Thus $\nabla\cdot\underset{\sim}{q} \equiv 0$ and, since the flow is steady, $\underset{\sim}{q}\cdot\nabla\rho \equiv 0$. It follows that $\nabla\cdot(\rho^{1/2}\underset{\sim}{q}) \equiv 0$, so there is a pseudo-stream-function ψ such that

$$(\psi_y, -\psi_x) = \rho^{1/2}\underset{\sim}{q} , \tag{2.1}$$

and it may be supposed, by way of normalization, that $\psi(x,0) \equiv 0$. It follows readily that $\rho = \rho(\psi)$ is a function of ψ alone. Note that in the special case of a parallel base flow $\underset{\sim}{q} = (W(y),0)$, the associated normalized pseudo-stream-function is

$$\Psi(y) = \int_0^y \rho^{1/2}(z)W(z)dz , \tag{2.2}$$

with ρ being the specified function of height above the bottom boundary.

The Euler equations for steady flow are

$$\rho(\underset{\sim}{q}\cdot\nabla)\underset{\sim}{q} = -\nabla p - g\rho\underset{\sim}{k} , \tag{2.3}$$

where p is the pressure, g the gravity constant, and $\underset{\sim}{k}$ the unit vector $(0,1)$. Keeping in mind (2.1) and the dependence of ρ solely on ψ, and introducing $\eta = \Delta\psi$, we easily derive from (2.3) the relations,

$$\eta\psi_x = \partial_x(p + \tfrac{1}{2}\rho|\underset{\sim}{q}|^2) ,$$

and (2.4)

$$\eta\psi_y = \partial_y(p + \tfrac{1}{2}\rho|\underset{\sim}{q}|^2) + g\rho .$$

Let $H = p + \frac{1}{2}\rho|\underset{\sim}{q}|^2 + g\rho y$ be the <u>total</u> <u>head</u> associated to our steady flow. Now H is also a function of ψ alone, as follows immediately from (2.4) and the fact that $\rho = \rho(\psi)$. If H is differentiated with respect to ψ using the chain rule, and the result simplified by employing the relations (2.4),

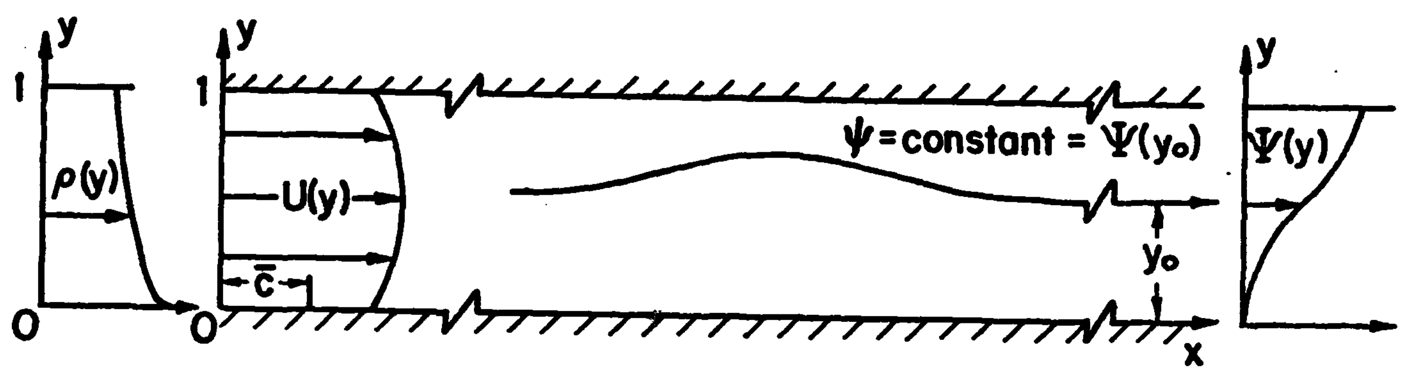


FIGURE 1. Definition sketch showing the primary density and velocity distributions $\rho(y)$ and $U(y)$, respectively. The pseudo-stream-function ψ is a function of the height y above the bottom boundary and the horizontal coordinate x.

there appears

$$-\Delta\psi + H'(\psi) = gy\rho'(\psi) , \tag{2.5}$$

where the ' denotes differentiation with respect to ψ. This is Yih's version of Long's equation (cf. Dubreil-Jacotin 1935, Long 1953 and Yih 1958). The auxillary stipulations are the kinematical conditions,

$$\psi(x,0) = \Psi(0) = 0 \quad \text{and} \quad \psi(x,1) = \Psi(1) , \tag{2.6}$$

for $x \in \mathbb{R}$, and the asymptotic conditions,

$$\psi(x,y) \to \Psi(y), \quad \text{as} \quad |x| \to \infty , \tag{2.7}$$

for $0 < y < 1$. Thus ψ represents a flow that is connected to the primary state at infinity, and for which ρ and H are constant along its pseudo-streamlines. The situation envisaged is illustrated in figure 1.

In (2.5) neither ρ nor H is immediately known as a function of ψ. Because of (2.7), they may be determined in principle from the primary flow. For example, if $U(y) \equiv d$, a constant, and $c = d-\bar{c}$, then (2.2) yields

$$\Psi(y) = c \int_0^y \rho^{1/2}(z)dz .$$

Since $\rho > 0$, Ψ is strictly increasing and so may be inverted. Let $Y = \Psi^{-1}$. Then the density ρ is expressed as a function of the pseudo-stream-function value Z by

$$\rho(Z) = \rho(Y(Z)) .$$

Since ρ is constant along pseudo-stream-lines, its value at a point (x_0,y_0), in the flow corresponding to ψ, is determined by tracing the pseudo-steamline with value $\psi(x_0,y_0)$ to infinity, where, because of (2.7), ρ is determined by the last displayed relation. Note that once $\rho(\psi)$ is known, H is determined from (2.1) and the hydrostatic pressure law $dp/dy = -g\rho$ for the primary flow.

Define the perturbed pseudo-stream-function $\phi = \phi(x,y)$ by the formula,

$$\psi(x,y) = \Psi(y) + j\phi(x,y) , \tag{2.8}$$

where j is a normalizing constant. The boundary-value problem expressed in (2.5, 2.6, 2.7) reduces to the nonlinear eigenvalue problem,

$$-\Delta\phi + h(y,\phi) = \lambda f(y,\phi) \qquad (2.9a)$$

with the supplementary conditions

$$\text{and} \quad \left.\begin{array}{l} \phi(x,0) = \phi(x,1) = 0, \quad \text{for} \quad x \in \mathbf{R}, \\ \phi(x,y) \to 0, \quad \text{as} \quad |x| \to \infty, \end{array}\right\} \qquad (2.9b)$$

for all y in $[0,1]$. The eigenvalue λ is proportional to gc^{-2}, where c is a velocity scale for the primary flow in the travelling coordinates introduced earlier (e.g. $c = \sup_{0\leq y\leq 1} W(y)$). Just as for $\rho(\psi)$ and $H(\psi)$, the functions $h(y,\phi)$ and $f(y,\phi)$ are in general a little complicated to determine from a given base state, and this aspect is ignored here. It is dealt with in detail in section 6, where a set of examples will be exhibited and shown to fall within the class of equations to which our theory is applicable.

The remainder of this section is devoted to various aspects of the boundary-value problem (2.9) and its set of solutions. First, by the term solitary wave we will understand a solution ϕ of (2.9) which is even in x, monotone for $x > 0$, and rapidly convergent to zero as $|x| \to \infty$. The associated pseudo-stream-function $\psi = \Psi + j\phi$ represents a steady flow, as described previously, which is a single-crested wave of translation, symmetric about its crest, and decaying rapidly to the base flow away from its crest, thus being in essence a disturbance of finite extent. The description of such flows as solitary waves is provisional, however. In order that such waves fall within the presently accepted conception of solitary waves (cf. Scott Russell 1844 and Miura 1976) their stability must be ascertained.

Similarly, solutions ϕ of (2.9a) which vanish at $y = 0$ and $y = 1$, and which are even periodic functions of x, say of period $2k$, and which are monotone on $[0,k]$ will be referred to as internal cnoidal waves. The name is chosen in analogy with the classical two-dimensional waves on the surface of a liquid (cf. Kortweg and de Vries 1895). For it is imagined that the flow corresponding to such a ϕ is a periodic wavetrain, propagating steadily and without change of form, and single-crested within its fundamental period. The reader is cautioned that this interpretation cannot be strictly justified. For, according to the derivation of Yih's equation (2.5), unless the pseudo-streamlines connect asymptotically to those of the base flow (i.e. $\phi \to 0$ as $|x| \to \infty$), the functions ρ and H cannot be evaluated by passing far upstream as suggested earlier (though we will see that for large periods, the flow in the trough agrees essentially with the primary flow). Consequently, whilst a periodic solution ϕ of (2.9a) does indeed define a solution of the Euler equations (2.3), it may not correspond to a perturbation of the given base flow. Its direct hydrodynamical interpretation thus becomes clouded. A more subtle interpretation of the generation and role of such periodic internal wavetrains has been proposed by Benjamin (1966), and this possibility receives some attention in section 7.

The issue of whether or not the pseudo-streamlines each connect to a definite value at infinity, as in figure 1, arises even for the solitary-wave solutions of (2.9). For solutions of (2.9a) which satisfy the boundary condition (2.9b) might conceivably take values strictly outside the range $[0,\Psi(1)]$. If this occurs, there would necessarily be closed streamlines in the finite region of the flow domain, representing a motion with an eddy of fluid not present in the base flow. Indeed, closed eddies, often called "rotors' do appear in practice, associated with larger amplitude waves, as in

the experimental observations of Davis and Acrivos (1967). It is plausible that the fluid encompassed within a rotor is nearly homogeneous, reflecting the mixing that occurs in such a structure. One might therefore be tempted to model this situation by extending the functions ρ and H outside the domain $[0,\Psi(1)]$ with the specification, $\rho(x) = \rho(0)$, for $\psi < 0$, and $\rho(x) = \rho(\Psi(1))$, for $\psi > \Psi(1)$, and similarly for H (cf. Benjamin 1971). In view of the serious limitations of perfect-fluid theory in describing this sort of flow feature, some circumspection regarding such eddies seems warranted. Accordingly, in section 5 a theorem is proved which may be interpreted as stating that closed eddies do not occur in a range of the flows whose existence is established by our theory.

The final point respective of (2.9) concerns the possibility of trivial solutions of (2.9a). The function $\phi \equiv 0$ is a solution of (2.9) that corresponds to the undisturbed base flow. As it happens, there is generally another trivial solution called the conjugate flow (cf. Benjamin 1971). This is the unique, non-zero, non-negative x-independent solution of (2.9a) that respects the first boundary condition in (2.9b). The conjugate flow plays an important role in the interpretation of various wave phenomena. However, as the reader will readily appreciate, these two trivial solutions, the base flow and its conjugate flow, potentially complicate a search for non-trivially x-dependent solutions of (2.9) that are of especial interest here.

3. EXISTENCE OF PERIODIC WAVES

In this section we prove the existence of solutions (λ, ϕ) of the boundary value problem,

$$\left.\begin{aligned} -\Delta\phi(x,y) + h(y,\phi(x,y)) &= \lambda f(y,\phi(x,y)) \quad \text{in } \Omega , \\ \phi(x,0) = \phi(x,1) &= 0 , \end{aligned}\right\} \quad (P)(3.1)$$

which are periodic in x, where $\Omega = \{(x,y) | -\infty < x < \infty, 0 < y < 1\}$. In the problem (3.1) we assume that f and h are defined for all values of ϕ. As we ultimately show that solutions ϕ of (3.1) determine pseudo-stream-functions ψ whose values lie in the physical range $[0, \Psi(1)]$ (cf. corollary 4.5) this embodies no essential restriction. The basic assumptions on f and h are contained in

Condition A. The function f has the form,

AI $$f(y,t) = \begin{cases} tf_0(y) + f_1(y,t), & \text{for } 0 \le y \le 1, t \ge 0 \\ -f(y, -t), & \text{for } 0 \le y \le 1, t < 0 , \end{cases}$$

where f_0, f_1 are Hölder continuous on bounded sets; $f_0 > 0$ on $0 < y < 1$; and $f_1(y,t) = o(t)$, uniformly for $0 \le y \le 1$, as $t \to 0$. Further, there exist constants $m > 1$, $n > 1$, $\alpha > 0$ and $d > 0$ such that for $t \ge 0$

$$\alpha t^m \le f_1(y,t) \le d(1 + t^n) .$$

The function h has the form,

AII $$h(y,t) = \begin{cases} th_0(y) + h_1(y,t), & \text{for } 0 \le y \le 1, t \ge 0 , \\ -h(y, - t), & \text{for } 0 \le y \le 1, t < 0 , \end{cases}$$

where h_0 and h_1 are Hölder continuous on bounded sets, $\frac{\partial h}{\partial t}$ is continuous and $\frac{\partial h}{\partial t} \geq e$ for some constant $e > -\pi^2$. Further there are constants $\sigma > 0$ and σ' such that for $t \geq 0$,

$$\sigma' t^m \leq h_1(y,t) \leq \sigma t^m,$$

and

$$|h_1(y,t)| \leq d(1 + t^n),$$

where m, n, and d are the constants appearing in AI. Note that the first inequality above implies $h_1(y,t) = o(t)$, uniformly for $0 \leq y \leq 1$, as $t \to 0$.

The final assumption and the subsequent variational principles involve the functions

$$\begin{aligned} F(y,t) &= 2 \int_0^t f(y,s)ds, \quad F_1(y,t) = 2 \int_0^t f_1(y,s)ds \\ H(y,t) &= 2 \int_0^t h(y,s)ds, \quad H_1(y,t) = 2 \int_0^t h_1(y,s)ds \quad . \end{aligned} \tag{3.2}$$

The last hypothesis is,

AIII For each $\lambda > \alpha^{-1}\sigma$ (α and σ the constants in AI and AII) there is a $\theta < 1$ such that

$$\lambda F_1(y,t) - H_1(y,t) \leq \theta(\lambda f_1(y,t)t - h_1(y,t)t) \ .$$

While the assumptions are rather technical, they are tailored with two ends in view. First they are weak enough to be readily verified in a range of applications. Further, they are strong enough to mitigate certain technical difficulties associated with our proofs.

The problem before us is to find solutions of problem (P) which are periodic in x with period $2k$. We consider two formulations of the problem. The first is a constrained problem (PC):

$$\left.\begin{array}{l} \text{solve} \quad (P) \\ \text{subject to} \quad \int_0^1 \int_{-k}^{k} [|\nabla\phi|^2 + H(y,\phi)]\,dxdy = R^2, \\ \text{where } R \text{ is a given positive constant.} \end{array}\right\} \quad (PC)(3.3)$$

The second is a free problem (PF):

$$\left.\begin{array}{c} \text{solve} \quad (P) \\ \text{where } \lambda \text{ is a given constant.} \end{array}\right\} \quad (PF)(3.4)$$

The problem (PC) corresponds to specifying the "energy" of a wave, while (PF) corresponds to specifying its velocity.

The analysis of both problems (PC) and (PF) is based on variational methods and the theory of rearrangements of functions. For each problem it will be shown that there is a critical point ϕ of a suitable functional and that the function ϕ is a solution of a weak formulation of the problem. Moreover, ϕ can be taken to be even in x, *nonnegative, and nonincreasing* on $0 < x < k$ for each y. That is, ϕ is symmetrized, according to the following definition.

Definition 3.1. Let $\phi = \phi(x,y)$ be continuous on $[-k,k] \times [0,1]$ and for each y let $\mu(\phi,a,y)$ denote the Lebesgue measure of the set $\{x \mid \phi(x,y) > a\}$. A function $\hat{\phi}$ which is even in x, nonnegative, nonincreasing on $0 < x < k$ for each y and satisfies

$$\mu(\hat{\phi},a,y) = \mu(|\phi|,a,y)$$

is called the symmetrization of ϕ. If $\phi = \hat{\phi}$ we call it symmetrized.

The construction of $\hat{\phi}$ and the properties of $\hat{\phi}$ that we use are given in an appendix. While the use of symmetrized functions is not needed for the existence of periodic solutions, it plays a central role in the estimates in section 4 which we use to show that a solitary wave is obtainable as the limit of periodic waves with increasing period. A symmetrized function, moreover, has an obvious appeal in that observed permanent waves exhibit such a form.

We note that there are solutions of (P) of any period, namely the zero solution and a countably infinite collection of non-zero x-independent solutions obtainable from bifurcation theory (Crandall and Rabinowitz 1970, Turner 1973). One of this latter class of solutions, a positive one, is the conjugate flow studied by Benjamin (1971). We shall show in section 4 that the solutions we obtain here are x-dependent provided k is sufficiently large.

The analysis will be carried out principally in the Hilbert space $H_k = H_k(\mathbb{R} \times (0,1))$ defined as follows: let C_k^∞ be the space of C^∞ functions which have support where $0 < y < 1$ and which are periodic in x with period $2k$. Define

$$\|u\|_k^2 = \iint_{\Omega_k} \{|\nabla u|^2 + h_0(y)u^2\}\,dxdy\ , \tag{3.5}$$

where

$$\Omega_k = \{(x,y) : |x| < k,\ 0 < y < 1\}\ . \tag{3.6}$$

The Poincare inequality,

$$\iint_{\Omega_k} |\nabla u|^2 dxdy \geq \pi^2 \iint_{\Omega_k} u^2 dxdy\ , \tag{3.7}$$

together with the inequality $h_0(y) \geq e > -\pi^2$, a consequence of hypothesis AII, show that $\|\ \|_k$ is, in fact, a norm. We let H_k be the completion of C_k^∞ in the norm (3.5). Then H_k is a Hilbert space and the corresponding

inner product is

$$(u,v)_k = \iint_{\Omega_k} \{\nabla u \cdot \nabla v + h_0(y)uv\} dxdy \ . \tag{3.8}$$

If ℓ is a continuous linear functional on H_k, its value at u is denoted by $\langle \ell, u \rangle$.

We will also have occasion to use the spaces C^j, $C^{j,\omega}$, L^p, and $W^{j,p}$ for various domains (cf. Adams 1975, Gilbarg and Trudinger 1977). A further consequence of the Poincaré inequality (3.7) is that for $u \in H_k$,

$$\|u\|_k \le C_1 \iint_{\Omega_k} |\nabla u|^2 dxdy \le C_2 \|u\|_{W^{1,2}(\Omega_k)} \le C_3 \|u\|_k \ , \tag{3.9}$$

with C_1, C_2, and C_3 independent of k.

In what follows we use the letter C, with or without subscripts or superscripts, to denote a positive constant which may be different in different inequalities and which may depend upon pertinent parameters, but which is independent of k. Moreover, we will often use the convention

$$\int g = \iint_{\Omega_k} g(x,y) dxdy \ . \tag{3.10}$$

Combining (Adams 1975, lemma 5.14) with (3.9) one finds that H_k is compactly embedded in $L^p(\Omega_k)$, $2 \le p < \infty$ and that

$$\|u\|_{L^p(\Omega_k)} \le C \|u\|_k , \ 2 \le p < \infty \ . \tag{3.11}$$

If $u \in H_k \cap W^{2,2}(\Omega_k)$ and $v \in H_k$, then

$$(u,v)_k = \int (-\Delta u + h_0(y)u)v \tag{3.12}$$

the boundary terms on $x = \pm k$ cancelling because of periodicity. We are thus led to define a weak periodic solution of (P) as a pair (λ, ϕ), $\phi \in H_k$, satisfying

$$(\phi, v)_k + \int h_1(y,\phi)v = \lambda \int f(y,\phi)v \tag{3.13}$$

for all $v \in H_k$. The integrals in (3.13) exist by virture of (3.11) and assumptions AI and AII. Weak solutions of problem (P) will be obtained from variational principles suited to the separate cases (PC) and (PF).

Consider first the problem (PC) and adopt the notation (cf. 3.2), that for a function u defined on Ω_k,

$$\left.\begin{aligned} I(u) &= \int\{|\nabla u|^2 + H(y,u)\}, \\ J(u) &= \int F(y,u), \\ \text{and for } R > 0, \qquad & \\ S(R) &= \{u \in H_k : I(u) = R^2\}. \end{aligned}\right\} \tag{3.14}$$

The assumptions on f and h guarantee that the functionals I and J are defined on H_k. One calls ϕ a critical point of J on $S(R)$ if the derivative of J in directions tangent to $S(R)$ is zero; that is, the derivative $J'(\phi)$ is parallel to $I'(\phi)$. A consequence of this is equation (3.13).

Theorem 3.2. Suppose f and h satisfy condition A. Then for each $k > 0$ the problem (PC)(3.3) has a solution (λ_k,ϕ_k), $\phi_k \in H_k \cap C^2(\Omega_k)$, which satisfies

1) $J(\phi_k) = \sup\limits_{u \in S(R)} J(u)$,

2) $\lambda_k > 0$ and $\phi_k > 0$ in Ω_k,

3) $\phi_k = \hat{\phi}_k$ (cf. definition 3.1).

Proof. We fix k and suppress it in writing λ and ϕ. One readily verifies that I and J are continuously differentiable and that, for u, $v \in H_k$,

$$\langle I'(u),v\rangle = 2\int(\nabla u\cdot\nabla v + h(y,u)v),$$

$$\langle J'(u),v\rangle = 2\int f(y,u)v.$$

Moreover, I' is strongly monotone. For, using the Poincaré inequality (3.7), we see that

$$\begin{aligned}\langle I'(u) - I'(w),u-w\rangle &= 2\int\{|\nabla(u-w)|^2 + (h(y,u) - h(y,w))(u-w)\}\\ &\geq 2\int\{|\nabla(u-w)|^2 + e(u-w)^2\}\\ &\geq 2(1 + \frac{\tilde{e}}{\pi^2})\int|\nabla(u-w)|^2,\end{aligned}$$

where $\tilde{e} = \min(e,0)$. Since $1 + \tilde{e}/\pi^2 > 0$, according to hypothesis AII, it follows from (3.9) that

$$\langle I'(u) - I'(w), u-w\rangle \geq C\|u-w\|_k^2. \tag{3.15}$$

Then from the standard relationship

$$I(u) = \int_0^1 \langle I'(\tau u),u\rangle d\tau$$

and (3.15) one has

$$\begin{aligned}I(u) &\geq C\int_0^1 \tau\|u\|_k^2 d\tau\\ &= \frac{C}{2}\|u\|_k^2.\end{aligned} \tag{3.16}$$

Thus all functions in a level set $S(R)$ of I are uniformly bounded in H_k. Then because of (3.11) and the growth conditions required of f in hypothesis AI, for each $R > 0$, J, in turn, is uniformly bounded on $S(R)$. For a fixed $R > 0$, let

$$\sup_{u\in S(R)} J(u) = b \quad .$$

Assumption AI implies further that $J(u) = 0$ if and only if $u = 0$, so $b > 0$.

The continuous functions which are piecewise linear in x and y and have nonzero partial derivatives with respect to x a.e., form a dense linear subspace of H_k (cf. the appendix), so there is a sequence of such piecewise-linear functions $u_j \in S(R)$, $j = 1,2,\ldots,$ for which $J(u_j) \to b$ as $j \to \infty$. Letting $\hat{u}_j$ be the symmetrization of u_j, we see from relation (Ap2) of the appendix that $J(\hat{u}_j) = J(u_j)$ for $j = 1,2,\ldots,$ while from (Ap2) and (Ap3) $I(\hat{u}_j) \leq I(u_j) = R^2$. Since $\|\hat{u}_j\|_k$ is uniformly bounded, according to (3.16), $\hat{u}_j$ has a subsequence converging weakly to a function $\phi \in H_k$. We will show ϕ is a solution of (PC).

As was noted in conjuction with inequality (3.11), H_k is compactly embedded in $L^{n+1}(\Omega_k)$, where n occurs in hypothesis AII, so that the subsequence converges strongly to ϕ in $L^{n+1}(\Omega_k)$. Moreover, a further subsequence of $\{\hat{u}_j\}$ may be assumed to converge pointwise a.e. to ϕ on Ω_k. According to (Krasnoselskii 1963, p. 30) the map $u \to F(y,u)$ is continuous from $L^{n+1}(\Omega_k)$ into $L^1(\Omega_k)$. Thus $J(\phi) = b$, and as $b > 0$, $\phi \not\equiv 0$. The functional I is the sum of the norm in H_k, which is lower semi-continuous, and the term $\int H_1(y,u)$ which, like $J(u)$, is continuous. Hence $I(\phi) \leq R^2$. Suppose that $I(\phi) < R^2$. Since I' is strongly monotone, the function $I(t\phi)$ is increasing in t and approaches $+\infty$ as $t \to +\infty$. Since $I(t\phi)$ is also continuous in t, there is a $t > 1$ for which $I(t\phi) = R^2$. Hypothesis AI implies that $F(y,u)$ is strictly increasing in the variable u, so $J(t\phi) > b$. This contradicts the characterization of b and hence $I(\phi) = R^2$. Since $\langle I'(\phi),\phi\rangle > 0$ we can use a lemma of Lyusternik (Vainberg 1964, p. 96) to conclude that $J'(\phi)$ is a multiple of $I'(\phi)$. Since $J'(\phi) \neq 0$ we can write this as

$$I'(\phi) = \lambda J'(\phi),$$

and this is tantamount to (3.13).

It follows from hypothesis AI that $\langle J'(\phi),\phi\rangle > 0$ and hence $\lambda > 0$. As regards ϕ, since it is a weak solution of an elliptic equation, it follows from the L^p elliptic theory (Agmon, Douglis and Nirenberg 1959) and inequality (3.11) that ϕ is in $W^{2,p}(\Omega_k)$ for any $p < \infty$. The Sobolev embedding theorem shows that $\phi \in C^{1,\alpha}(\Omega_k)$ and the Schauder theory (Gilbarg and Trudinger 1977) yields $\phi \in C^{2,\alpha}(\overline{\Omega}_k)$. Being the pointwise limit of symmetrized functions, the continuous function ϕ inherits the property of being symmetrized; i.e. $\phi = \hat{\phi}$. To see that ϕ is strictly positive on Ω_k we use hypothesis AII to write $h(y,\phi(x,y)) = \tilde{h}(y,\phi(x,y))\phi(x,y)$, and then decompose $\tilde{h}$ as $h^+(x,y) - h^-(x,y)$ with $h^\pm$ being nonnegative functions of x and y. Since $\phi \geq 0$, $-\Delta\phi + h^+\phi = h^-\phi + \lambda f(y,\phi)$, a nonnegative function which is not identically zero. The strong maximum principle (Gilbarg and Trudinger 1977) implies that $\phi > 0$ in Ω_k, completing the proof.

We next turn to the problem (PF). For use here and in the sequel, denote the smallest eigenvalue of the linear problem,

$$\left.\begin{aligned} -\Delta u + h_0(y)u &= \lambda f_0(y)u, \\ u &\in H_k, \end{aligned}\right. \qquad (3.17)$$

by μ. One obtains the eigenfunctions of (3.17) by separation of variables and it is easily seen that μ has an associated eigenfunction $\xi(y)$, independent of x. It is the eigenfunction associated with the lowest eigenvalue of

$$\left.\begin{aligned} -\xi_{yy} + h_0(y)\xi &= \lambda f_0(y)\xi \\ \xi(0) = \xi(1) &= 0 \end{aligned}\right\} \qquad (3.18)$$

and is known to be positive (Ince 1927, p. 235). The existence and positivity of ξ also follow easily from the type of arguments just completed for the problem (PC).

It will be convenient to normalize ξ so that

$$\int_0^1 \{\xi_y^2 + h_0\xi^2\} = 1. \tag{3.19}$$

For fixed λ define a functional M on H_k by

$$M(u) = \|u\|_k^2 + \int_{\Omega_k} \{H_1(y,u) - \lambda F(y,u)\}. \tag{3.20}$$

We use $\|M'(u)\|$ to denote the norm of the derivative $M'(u)$ as a functional on H_k. One readily verifies that a critical point of M, i.e. a point ϕ for which $\langle M'(\phi),v\rangle = 0$ for all $v \in H_k$, is a weak solution of problem (PF) (cf. 3.13). To show there is a critical point we use a slight modification of a result of Ambrosetti and Rabinowitz (1973, theorem 2.1).

<u>Proposition</u> 3.3. Let $k > 0$ be fixed and let M be a continuously differentiable functional on H_k satisfying the following conditions.

1) There are constants $a > 0$ and $r > 0$ such that $M(u) \geq a$ for $\|u\|_k = r$.
2) $M(0) = 0$ and there is a function w with $\|w\|_k > r$ such that $M(w) \leq 0$.
3) For each $\beta > 0$ if $\{u_i\}_{i=1}^\infty$ is a sequence satisfying $\beta < M(u_i) < \beta^{-1}$, for all i, and $\|M'(u_i)\| \to 0$, as $i \to \infty$, then a subsequence $\{u_{i_m}\}_{m=1}^\infty$ of $\{u_i\}_{i=1}^\infty$ converges strongly in H_k as $m \to \infty$.

Let

$$\Gamma = \{\gamma \in C([0,1],H_k) \mid \gamma(0) = 0,\ \gamma(1) = w\}$$

and define

$$\left.\begin{aligned} b_\gamma = b_\gamma(M) &= \max_{u\in\gamma([0,1])} M(u) \\ b &= \inf_{\gamma\in\Gamma} b_\gamma. \end{aligned}\right\} \tag{3.21}$$

Suppose γ_n, $n = 1,2,3,\ldots,$ is a sequence of paths in Γ such that

$$b_{\gamma_n} \leq b + \frac{1}{n}.$$

Then there is a subsequence $\{n_j\}_{j=1}^{\infty}$ of the positive integers and elements $u_{n_j} \in \gamma_{n_j}([0,1])$ such that u_{n_j} converges strongly to u as $j \to \infty$, $M(u) = b$, and $M'(u) = 0$.

Proof. For a real number z introduce

$$A_z = \{u \in H_k \mid M(u) \leq z\}$$

and

$$K_z = \{u \in H_k \mid M(u) = z \text{ and } M'(u) = 0\}.$$

For $j = 1,2,3,\ldots$ let

$$N_j = \{u \in A_{b+\frac{1}{j}} \setminus A_{b-\frac{1}{j}} \mid \|M'(u)\| < \tfrac{1}{j}\}.$$

Then for each j, N_j contains a neighborhood of K_b. From (Ambrosetti and Rabinowitz 1973, lemma 1.3) it follows that for each j there is a deformation map $\eta_t^j(x)$, taking $(t,x) \in [0,1] \times H_k$ continuously into H_k and constants $\bar{\varepsilon}_j > \varepsilon_j > 0$ such that

1) $\eta_0^j(u) = u$ for $u \in H_k$,

2) $\eta_t^j(u) = u$ for $u \in A_{b-\bar{\varepsilon}_j}$,

3) $\eta_1^j(A_{b+\varepsilon_j} \setminus N_j) \subset A_{b-\varepsilon_j}$.

For each j choose a positive integer m_j so that $m_j^{-1} < \varepsilon_j$. If $\gamma_{m_j}([0,1]) \cap N_j = \emptyset$, then since the composition $\eta_1^j \circ \gamma_{m_j}$ is a member of Γ and $M(u) < b-\varepsilon_j$ on the image of this composition, we have a contradiction to the characterization (3.21). Hence there is a function $u_{m_j} \in \gamma_{m_j}([0,1]) \cap N_j$ for each j. The compactness property (3) guarantees

that a subsequence $\{u_{n_j}\}$ of $\{u_{m_j}\}$ converges strongly as $j \to \infty$ to a function u, from which it follows that $M(u) = b$, and $M'(u) = 0$. Note that the hypotheses of the proposition imply that $b \geq a > 0$ and thus $u \neq 0$.

We can now show that problem (PF) has a symmetrized solution.

<u>Theorem</u> 3.4. Let f and h satisfy condition A with α and σ as defined therein. Let μ be the lowest eigenvalue of problem (3.17). If $\alpha^{-1}\sigma < \lambda < \mu$ then for each $k > 0$ the problem (PF)(3.4) has a solution $\phi_k \in H_k \cap C^2(\Omega)$ with the following properties.

1) $M(\phi_k) = \inf_{\gamma \in \Gamma} b_\gamma(M)$ (cf. 3.20 - 3.21),

2) $\phi_k > 0$ in Ω,

3) $\phi_k = \hat{\phi}_k$ (c.f. definition 3.1).

<u>Proof</u>. Fix $k > 0$. Since μ is the smallest eigenvalue of (3.17) it follows that

$$\|u\|_k^2 \geq \mu \int f_0(y)u^2 \qquad (3.22)$$

for any $u \in H_k$. Hypothesis AI implies that for any $\delta > 0$ there is a constant $C_\delta > 0$ so that

$$|f_1(y,t)| \leq \delta|t| + C_\delta|t^n|,$$

from which it follows that

$$\int F_1(y,u) \leq \int [\delta u^2 + C'_\delta|u|^{n+1}].$$

Upon invoking the embedding inequality (3.11) we find that

$$\int F_1(y,u) \leq (C\delta + C''_\delta \|u\|_k^{n-1})\|u\|_k^2,$$

so that the functional $\int F_1$ is $o(\|u\|_k^2)$ as $\|u\|_k \to 0$. One obtains a corresponding estimate with F_1 replaced by H_1. These estimates, with the inequality (3.22), lead to

$$M(u) \geq (1 - \frac{\lambda}{\mu})\|u\|_k^2 + o(\|u\|_k^2)$$

as $\|u\|_k \to 0$. So for each $\lambda < \mu$, M satisfies hypothesis (1) of proposition 3.3 for any sufficiently small positive r.

Plainly $M(0) = 0$. To find a function w such as occurs in hypothesis (2) of proposition 3.3, first observe that from AI and AII,

$$\lambda F_1(y,t) - H_1(y,t) \geq (\lambda\alpha-\sigma)t^{m+1}/(m+1).$$

Now if u is any positive, symmetrized function in H_k and $s > 0$, then

$$\int\{\lambda F_1(y,su) - H_1(y,su)\} \geq \frac{(\lambda\alpha-\sigma)s^{m+1}}{m+1}\int u^{m+1},$$

and so from (3.20),

$$M(su) \leq s^2(\|u\|_k^2 - \lambda\int f_0(y)u^2) - \frac{(\lambda\alpha-\sigma)s^{m+1}}{m+1}\int u^{m+1}.$$

Then, since $m > 1$, for s sufficiently large, say $s = s_0$, $M(s_0u) \leq 0$ and $\|s_0u\|_k > r$. We take $w = s_0u$ and have $w = \hat{w}$. That M is C^1 and satisfies hypothesis (3), often called the Palais-Smale condition, is shown as in (Ambrosetti and Rabinowitz 1973, lemma 3.6). Proposition 3.3 is now applicable and the existence of a solution of problem (PF) satisfying (1) can be concluded. However, establishing a solution satisfying (1), (2), and (3) requires further argument. To that end it will be verified that there is a sequence of paths $\gamma_n \in \Gamma$, $n = 1,2,3,\cdots$, with images consisting solely of symmetrized functions for which b_{γ_n} converges to the critical value b. Proposition 3.3 will then yield a critical point satisfying properties (1), (2), and (3).

We begin by showing that the functional M is Lipschitz continuous on bounded subsets of H_k. Let

$$G_1(y,t) = \lambda F(y,t) - H_1(y,t) \qquad (3.23)$$

and

$$N(u) = \int G_1(y,u).$$

Then from (3.20),

$$
\begin{aligned}
|M(u) - M(v)| &\le |\|u\|_k^2 - \|v\|_k^2| \\
&+ \left|\int_0^1 \langle N'(\tau(u-v) + v, u-v\rangle \, d\tau\right| \\
&= (\|u\|_k + \|v\|_k)\, |\|u\|_k - \|v\|_k| \\
&+ \left|\int_0^1 \int_{\Omega_k} [g_1(y,\tau(u-v) + v](u-v)d\tau\right| \\
&\le (\|u\|_k + \|v\|_k + C\int_0^1 \|g_1(y,\tau(u-v) + v)\|_{L^2(\Omega_k)} d\tau)\|u-v\|_k,
\end{aligned}
$$

where $g_1(y,t) = \partial_t G_1(y,t)$. The growth conditions in AI and AII imply that the expression in parentheses is bounded on any bounded subset of H_k.

Let γ be a path in Γ and suppose $\|\gamma(t)\|_k \le s$ for $0 \le t \le 1$. Let B_s denote the ball of radius s centered at 0 in H_k and let C denote the Lipschitz constant for M, associated with the ball B_{2s}. Given $\varepsilon > 0$ $(\varepsilon < 4Cs)$ choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ so that $\|\gamma(t_{i+1}) - \gamma(t_i)\|_k < \varepsilon/4C$ for $i = 0,1,\ldots n-1$. Let $u_0 = 0$, $u_n = w$, and for $i = 1,2,\cdots,n$ let u_i be a piecewise linear function of x and y having $\partial_x u_i \ne 0$ a.e., such that $\|u_i - \gamma(t_i)\|_k < \varepsilon/4C$ (cf. the appendix). Define a path $\tilde{\gamma}$ by linearly interpolating the functions u_i;

$$
\tilde{\gamma}(t) = (1 - (nt-i))u_i + (nt-i)u_{i+1}, \quad \frac{i}{n} \le t \le \frac{i+1}{n} \qquad (3.24)
$$
$$
i = 0,1,2,\cdots,n-1.
$$

Since $\varepsilon < 4Cs$ each u_i is in B_{2s} and hence the path $\tilde{\gamma}$ lies in B_{2s}. Using the Lipschitz property of M one checks that $b_{\tilde{\gamma}} \le b_\gamma + \varepsilon/2$. Suppose $\|u_i\|_{L^\infty(\Omega_k)} \le Q$ for $i = 0,1,\cdots,n$. Then for each $t \in [0,1]$, $\tilde{\gamma}(t)$ is a convex combination of some pair u_i, u_{i+1} and hence $\|\tilde{\gamma}(t)\|_{L^\infty} \le Q$.

It will now be convenient to let

$$
G(y,t) = \lambda F(y,t) - H(y,t)
$$

and express M as

$$M(u) = \int\{|\nabla u|^2 - G(y,u)\}.$$

Since $G(y,t)$ is uniformly continuous for $0 \leq t \leq Q$, there is an $\eta > 0$ so that if $|t-\tilde{t}| < \eta$, then $|G(y,t) - G(y,\tilde{t})| < \varepsilon/4k$, where $2k$ is the period. We suppose that $\|u_i - u_{i+1}\|_{L^\infty} < \eta$ for $i = 0,1,\ldots n-1$. If this is not the case, the t interval may be partitioned more finely so that at adjacent points of the partition the values of $\tilde{\gamma}(t)$ differ in L^∞ by at most η. Assuming that $\tilde{\gamma}$ is defined by (3.24) for a possibly finer partition, let $\hat{\gamma}$ be the path obtained by replacing each function u_i in (3.24) by its symmetrization $\hat{u}_i$. A convex combination of $\hat{u}_i$ and $\hat{u}_{i+1}$ will be symmetrized and so $\hat{\gamma}$ consists entirely of symmetrized functions. Note that w, the end of the original path, satisfies $w = \hat{w}$. The path $\hat{\gamma}$ is a substitute for the set one would get by symmetrizing each point $\gamma(t)$ on the path. Symmetrizing each point on a path does not obviously produce a path.

Now consider M on the path $\hat{\gamma}$. On the i^{th} interval of the partition used in (3.24), $\hat{\gamma}(t)$ may be expressed as $(1-\tau)\hat{u}_i + \tau\hat{u}_{i+1}$ where $\tau = nt - i$. By convexity,

$$\int|\nabla((1-\tau)\hat{u}_i + \tau\hat{u}_{i+1})|^2 \leq (1-\tau)\int|\nabla\hat{u}_i|^2 + \tau\int|\nabla u_{i+1}|^2.$$

Next, using inequality (Ap1) of the appendix it is concluded that

$$\|\hat{u}_i - \hat{u}_{i+1}\|_{L^\infty} \leq \|u_i - u_{i+1}\|_{L^\infty} < \eta,$$

and so

$$\begin{aligned}
\int|G(y,(1-\tau)\hat{u}_i &+ \tau\hat{u}_{i+1}) - (1-\tau)G(y,\hat{u}_i) - \tau G(y,\hat{u}_{i+1})| \\
&\leq (1-\tau)\int|G(y,(1-\tau)\hat{u}_i + \tau\hat{u}_{i+1}) - G(y,\hat{u}_i)| \\
&\quad + \tau\int|G(y,(1-\tau)\hat{u}_i + \tau\hat{u}_{i+1}) - G(y,\hat{u}_{i+1})| \\
&\leq \int\frac{\varepsilon}{4k} \\
&= \frac{\varepsilon}{2},
\end{aligned}$$

because of the uniform continuity of G. Finally, using relations (Ap2) and (Ap3) of the appendix we estimate M on the i^{th} segment of $\hat{\gamma}$ by

$$\begin{aligned}
M((1-\tau)\hat{u}_i + \tau\hat{u}_{i+1}) &\le (1-\tau)\int|\nabla\hat{u}_i|^2 + \tau\int|\nabla\hat{u}_{i+1}|^2 \\
&\quad - (1-\tau)\int G(y,\hat{u}_i) - \tau\int G(y,\hat{u}_{i+1}) + \frac{\varepsilon}{2} \\
&= (1-\tau)M(\hat{u}_i) + \tau M(\hat{u}_{i+1}) + \frac{\varepsilon}{2} \\
&\le (1-\tau)M(u_i) + \tau M(u_{i+1}) + \frac{\varepsilon}{2} \\
&\le b_{\tilde{\gamma}} + \frac{\varepsilon}{2} \\
&\le b_\gamma + \varepsilon.
\end{aligned}$$

As the inequality holds on each segment of $\hat{\gamma}$,

$$b_{\hat{\gamma}} \le b_\gamma + \varepsilon.$$

If, for any positive integer n, γ_n is a path for which $b_{\gamma_n} < b + \frac{1}{n}$, according to what we have just shown, there is a path γ_n consisting of symmetrized functions for which $b_{\gamma_n} < b + \frac{2}{n}$. Then from proposition 3.3 there is a sequence $\{v_j\}_{j=1}^{\infty}$ with $v_j = \hat{v}_j$, for all j, converging in H_k to a nonzero critical point ϕ_k of M. It now follows exactly as in the proof of theorem 3.2 that ϕ_k is in $C^{2,\alpha}(\bar{\Omega}_k)$, that $\phi_k = \hat{\phi}_k$ and that $\phi_k > 0$ in Ω_k.

<u>Remark</u>. The restriction $\lambda < \mu$ in theorem 3.4 stems from our ultimate interest in waves of supercritical speed which, as will be seen, persist in the limit of increasing periods. While one can show the existence of nontrivial periodic solutions of problem (P) having $\lambda > \mu$, using bifurcation theory, they are not useful for our present study. Assuming $\lambda < \mu$, the further restriction $\alpha^{-1}\sigma < \lambda$ is essential as one can see from the example $-\Delta\phi + \sigma\phi^3 = \lambda(\phi + \alpha\phi^3)$ in which $\mu = \pi^2$. Suppose there were a nontrivial solution (λ,ϕ) with $\lambda \le \alpha^{-1}\sigma$. Taking the inner product of each side of the equation with ϕ one finds that $\int|\nabla\phi|^2 \le \lambda\int\phi^2$ and hence from (3.7), $\lambda \ge \pi^2$, which is excluded by the restriction $\lambda < \mu$.

One can see from the proof of theorem 3.4 that it is the behavior of the whole term $\lambda f_1 - h_1$ that is important in problem (PF). Thus, conditions AI and AII could, in this problem, be replaced by an analogous condition on $\lambda f - h$, so allowing for wider applicability of the theory.

4. BOUNDS FOR PERIODIC SOLUTIONS

The subsequent analysis depends on bounds on the solutions (λ_k, ϕ_k) of (3.1) whose existence was established in section 3. In the problem (PC)(3.3) the size of $\|\phi\|_k$ was specified while in the problem (PF)(3.4) λ_k was specified in the range $\alpha^{-1}\sigma < \lambda_k < \mu$. It will be important to have estimates independent of k for both λ_k and ϕ_k in the two problems. Such bounds will be obtained in the present section, using some additional hypotheses which are not unduly restrictive for the physical applications.

It will first be shown that $\lambda_k < \mu$ in the problem (PC). When it is convenient the subscript k will be suppressed.

<u>Lemma</u> 4.1. Suppose that f and h satisfy condition A of section 3, with the additional restrictions $m < 5$ and $2\alpha^{-1}\sigma < \mu$ on the parameters. It follows that there are constants $\eta = \eta(R) > 0$ and $k_0 = k_0(R)$ such that if (λ_k, ϕ_k) is the solution of (PC) obtained in theorem 3.2, with $I(\phi_k) = R^2$, then

$$\lambda_k \leq (1-\eta)\mu, \tag{4.1}$$

for $k > k_0(R)$.

<u>Proof</u>. We suppress the variable y, writing $f_1(\phi)$ for $f_1(y,\phi)$, and so on. In the weak form (3.13) of the problem (3.1), let $v = \phi = \phi_k$ and use the notation (3.14) to express $\lambda = \lambda_k$ as

$$\lambda = \frac{\int\{|\nabla\phi|^2 + h_0\phi^2 + h_1(\phi)\phi\}}{\int\{f_0\phi^2 + f_1(\phi)\phi\}}$$

$$= \frac{R^2 + \int\{h_1(\phi)\phi - H_1(\phi)\}}{J(\phi) + \int\{f_1(\phi)\phi - F_1(\phi)\}} .$$

Since $J(u) \leq J(\phi)$ for an arbitrary $u \in H_k$ with $I(u) = R^2$, then

$$\lambda \leq \frac{R^2 + \int\{h_1(\phi)\phi - H_1(\phi)\}}{J(u) + \int\{f_1(\phi)\phi - F_1(\phi)\}}. \tag{4.2}$$

Take $u(x,y)$ to be the element of H_k defined on $-k \leq x \leq k$ by $u(x,y) = A(x)\xi(y)$, where ξ is the eigenfunction defined in (3.18). We show that A can be chosen so that $I(u) = R^2$ and so that, for $k \geq k_0$, the right-hand member of (4.2) is at most $(1 - \eta)$, where k_0 and η are positive numbers depending on R.

The normalization $I(u) = R^2$ entails having

$$R^2 = n_2 \int_{-k}^{k} A_x^2 + \int_{-k}^{k} A^2 + \int_{\Omega_k} H_1(u), \tag{4.3}$$

where, for $0 \leq p < \infty$,

$$n_p = \int_0^1 \xi^p dy. \tag{4.4}$$

Here we have used the normalization $\int_0^1\{\xi_y^2 + h_0\xi^2\} = 1$ given earlier (cf. 3.19). The normalization also implies $\int_0^1 f_0\xi^2 = \mu^{-1}$, and thus

$$J(u) = \mu^{-1}\{\int_{-k}^{k} A^2 + \mu\int_{\Omega_k} F_1(u)\}. \tag{4.5}$$

Combining (4.2), (4.3), and (4.5) one sees that the inequality (4.1) will hold provided

$$\begin{aligned} n_2 \int_{-k}^{k} A_x^2 + \eta \int_{-k}^{k} A^2 &\leq \int\{(1-\eta)\mu F_1(u) - H_1(u)\} \\ &+ \int\{(1-\eta)\mu(f_1(\phi)\phi - F_1(\phi)) - (h_1(\phi)\phi - H_1(\phi))\}. \end{aligned} \tag{4.6}$$

From condition AIII (in this instance $\theta \leq 1$ suffices) it can be seen that

$$\begin{aligned} (1-\eta)\mu(f_1(\phi)\phi - F_1(\phi)) &- (h_1(\phi)\phi - H_1(\phi)) \\ &\geq (1-\eta)\mu f_1(\phi)\phi - h_1(\phi)\phi \\ &\qquad - \theta((1-\eta)\mu f_1(\phi)\phi - h_1(\phi)\phi) \\ &= (1-\theta)((1-\eta)\mu f_1(\phi)\phi - h_1(\phi)\phi) \end{aligned}$$

$$> (1-\theta)\{(1-\eta)\mu\alpha - \sigma\}\phi^{m+1}.$$

Hence it suffices to find an η with $0 < \eta < 1/2$ so that,

$$n_2 \int_{-k}^{k} A_x^2 + \eta \int_{-k}^{k} A^2 < \int [(1-\eta)\mu\ F_1(u) - H_1(u)] \tag{4.7}$$

for some u satisfying (4.3).

To continue, let $A(x) = R\beta g(\nu x)$ where $g(t) = e^{-2|t|}$ and where β and ν are constants to be determined. A short computation brings (4.3) to the form,

$$1 = 4n_2\nu\beta^2 c_2(\nu k) + \frac{\beta^2}{\nu} c_2(\nu k) + \delta R^{m-1} \frac{\beta^{m+1}}{\nu} c_{m+1}(\nu k), \tag{4.8}$$

where $c_m(s) = \int_{-s}^{s} g^m(t)dt$ and where $\delta = \delta(k)$ is a constant satisfying $-2\sigma' n_{m+1} < (m+1)\delta < 2\sigma n_{m+1}$ (cf. condition AII in section 3). Using conditions AI and AII one sees that (4.7) will hold provided

$$4n_2\nu\beta^2 c_2(\nu k) + \eta \frac{\beta^2}{\nu} c_2(\nu k) < 2n_{m+1} \frac{\gamma R^{m-1}\beta^{m+1}}{(m+1)\nu} c_{m+1}(\nu k), \tag{4.9}$$

where $\gamma = (1-\eta)\mu\alpha - \sigma > 0$. If we now let $\nu = \beta^{2+\varepsilon}$ and require η to satisfy the further condition $\eta < n_2\beta^{4+2\varepsilon}$ then (4.8) and (4.9) will follow from

$$1 = 4n_2\beta^{4+\varepsilon}c_2 + \beta^{-\varepsilon}c_2 + \delta R^{m-1}\beta^{m-1-\varepsilon}c_{m+1}, \tag{4.10}$$

and

$$5n_2\beta^{4+\varepsilon}c_2 < 2n_{m+1} \frac{\gamma R^{m-1}\beta^{m-1-\varepsilon}c_{m+1}}{m+1}, \tag{4.11}$$

respectively. Note that $c_p (p = 2, m+1)$ now has $\beta^{2+\varepsilon}k$ as its argument and further, that as the argument increases to infinity, c_p increases monotonically to $1/p$. We will henceforth assume that $k > k(\beta) = (\frac{1}{4}\ln 2)\beta^{-2-\varepsilon}$ so that $1/2p < c_p < 1/p$.

To satisfy (4.11) it will suffice to have $\beta < \beta_1(R)$ where

$$\beta_1(R) = \left(\frac{2n_{m+1}\gamma R^{m-1}}{5n_2(m+1)^2}\right)^{\frac{1}{5-m+2\varepsilon}}. \tag{4.12}$$

Here we've used the bound $c_{m+1}/c_2 > 1/(m+1)$. To satisfy (4.10), begin by choosing an ε_0 with $0 < \varepsilon_0 < m-1$ and letting $\beta_0 = 4^{-1/\varepsilon_0}$, so that $\beta_0^{-\varepsilon_0} c_2 > 1$. Our choice for β is

$$\beta(R) = \inf_{0<\varepsilon<\varepsilon_0} \min \left\{ \left(\frac{1}{8n_2}\right)^{\frac{1}{4+\varepsilon}}, \left(\frac{(m+1)^2}{8\sigma n_{m+1} R^{m-1}}\right)^{\frac{1}{m-1-\varepsilon}}, \beta_0, \beta_1(R) \right\}.$$

We further choose

$$k_0(R) = \left(\frac{1}{4}\ln 2\right)(\beta(R))^{-2-\varepsilon_0}$$

and

$$\eta(R) = \min\{\tfrac{1}{2}, n_2(\beta(R))^{4+2\varepsilon_0}\}.$$

Since $\beta_0 = 4^{-1/\varepsilon_0} < 1$, $\beta < \beta_0 < 1$ and so for all ε in $[0,\varepsilon_0]$, $\eta < n_2\beta^{4+2\varepsilon}$, as required. Assume $k > k_0$. Then for ε in $[0,\varepsilon_0]$, $k > k_0 > k(\beta)$, from above, so $1/2p < c_p < 1/p$.

Since $\beta(R) < \beta_1(R)$, (4.11) is satisfied. By the choice of $\beta(R)$ we also guarantee that the first and third terms in the right-hand side of (4.10) are at most 1/4 for any ε in $[0,\varepsilon_0]$. The remaining term, $\beta^{-\varepsilon}(R)c_2$ approaches $c_2 < 1/2$ as $\varepsilon \to 0$ and, since $\beta(R) < \beta_0$, exceeds 1 for $\varepsilon = \varepsilon_0$. By continuity there is an ε in $[0,\varepsilon_0]$ for which (4.10) is satisfied and we make it our choice. The trial function u is now fixed and the inequality (4.1) follows accordingly.

Remark. As R approaches 0, one has

$$\eta(R) = O(R^{\omega}), \quad \text{where } \omega = \frac{(m-1)(4+2\varepsilon_0)}{5-m}.$$

Lemma 4.2. Suppose that f and h satisfy condition A of section 1. Then there are constants C_1 and C_2 such that for any solution (λ,ϕ) of (PC)

with $\phi \in H_k$ and $I(\phi) = R^2 > 0$ one has $\lambda > 0$ and, moreover,

$$\lambda > \mu\left(\frac{1 - \sigma' C_1 R^{m-1}}{1 + \alpha C_2 R^{m-1}}\right). \tag{4.14}$$

Proof. As in the proof of lemma 4.1

$$\lambda = \frac{\|\phi\|_k^2 + \int h_1(\phi)\phi}{\int\{f_0\phi^2 + f_1(\phi)\phi\}}.$$

Since $\|\phi\|_k^2 > \mu\int f_0\phi^2$ by (3.22),

$$\lambda > \frac{1 + \|\phi\|_k^{-2}\int h_1(\phi)\phi}{\mu^{-1} + \|\phi\|_k^{-2}\int f_1(\phi)\phi},$$

and the desired estimate follows from the growth assumptions on f_1 and h_1 together with inequality (3.11).

We now have upper and lower bounds for λ occurring in the solution (λ,ϕ) of problem (PC) given in theorem 3.2. In the proof of that theorem the estimate $\|\phi\|_k < CR$ (cf. 3.16) was also derived. In the problem (PF) λ is restricted to the interval $(\alpha^{-1}\sigma,\mu)$ at the outset, in theorem 3.4. For this latter problem we next obtain a bound for $\|\phi\|_k$ in terms of λ.

Lemma 4.3. Suppose f and h satisfy condition A and further that $\alpha^{-1}\sigma < \lambda < \mu$ and $1 < m < 5$. Then the solution ϕ of problem (PF) obtained in theorem 3.4 satisfies

$$\|\phi\|_k < C\,\frac{(\mu-\lambda)^{\frac{5-m}{4(m-1)}}}{(\lambda\alpha - \sigma)}, \tag{4.15}$$

where C depends on θ and μ.

Proof. We will use the variational characterization of ϕ in theorem 3.4 to obtain (4.15). From proposition 3.3, for any path $\gamma \in \Gamma$,

$$\|\phi\|_k^2 + \int\{H_1(y,\phi) - \lambda F(y,\phi)\} = b \leq b_\gamma. \tag{4.16}$$

On the other hand the weak form (3.13) of the basic equation, with $v = \phi$, yields

$$\|\phi\|_k^2 + \int\{h_1(y,\phi)\phi - \lambda f_0\phi^2 - \lambda f_1(y,\phi)\phi\} = 0. \tag{4.17}$$

Combining (4.16) and (4.17), and using assumption AIII, one arrives at

$$(1-\theta)(\|\phi\|_k^2 - \lambda\int f_0(y)\phi\ )^2 \leq b_\gamma. \tag{4.18}$$

Now, by (3.22),

$$(1-\theta)\|\phi\|_k^2(1 - \frac{\lambda}{\mu}) \leq b_\gamma, \tag{4.19}$$

and this holds for any $\gamma \in \Gamma$.

We now construct a particular path γ_0 as follows. Let u be the function in H_k defined by,

$$u(x,y) = \exp\left(-(\mu-\lambda)^{1/2}|x|\right)\xi(y),$$

for $(x,y) \in [-k,k] \times [0,1]$, where, again, ξ denotes the eigenfunction associated with the lowest eigenvalue of (3.18), normalized as in (3.19). From the proof of theorem 3.4 there is inferred the existence of a $t_0 > 0$ such that, $M(t_0u) = 0$ where M is the functional defined in (3.20). Let γ_0 be the path defined on $0 \leq t \leq t_0$ by $\gamma_0(t) = tu$. A simple calculation shows that

$$M(tu) \leq \frac{t^2(\mu-\lambda)^{1/2}}{2}\left[\int_0^1 f_0\xi^2 + n_2\right] + (\sigma-\lambda\alpha)t^{m+1}n_{m+1}\frac{\left(1 - \exp[-2(m+1)k(\mu-\lambda)^{1/2}]\right)}{(m+1)^2(\mu-\lambda)^{1/2}}, \tag{4.20}$$

where n_p is defined in (4.4). By definition, the maximum of $M(tu)$, for t in the range $0 \leq t \leq t_0$ is b_{γ_0}, and another calculation, using (4.20),

shows that

$$b_{\gamma_0} < C_1 \frac{(\mu-\lambda)^{\frac{m+3}{2(m-1)}}}{(\lambda\alpha-\sigma)^{\frac{2}{m-1}}}. \tag{4.21}$$

The combination of (4.21) and (4.19) with $\gamma = \gamma_0$ produces the desired inequality (4.15).

As already noted, the smoothness of a solution ϕ follows from standard regularity theory. We repeat some of it here to exhibit estimates with constants independent of the period $2k$. In particular lemma 4.9 below will be used in showing the exponential decay of solutions in lemma 4.10.

Let $\eta = \eta(x)$ be an even C^∞ function supported on $-3/4 < x < 3/4$, taking values in $[0,1]$, and so chosen that $\eta(x) + \eta(1-x) = 1$ for $0 < x < 1$. This last property is equivalent to having $\eta(x) - 1/2$ odd with respect to $x = 1/2$ on the interval $0 < x < 1$. We omit a detailed construction of such a function. Given η we let $\eta_s(x) = \eta(x-s)$ for any real s and note that the collection η_j, j an integer, forms a partition of unity on the line. Let $\zeta = \zeta(x)$ stand for an arbitrary cutoff function, which is to say, ζ is a C^∞ function with range in $[0,1]$ having compact support. We use the estimate,

$$\|u\|_{W^{2,p}(\mathcal{D})} < C(\|\Delta u\|_{L^p(\mathcal{D})} + \|u\|_{L^p(\mathcal{D})} + \|u\|_{L^\infty(\partial\mathcal{D})}), \tag{4.22}$$

(Agmon, Douglis, Nirenberg 1959, thm. 15.2) valid for a domain $\mathcal{D}$ with smooth boundary $\partial\mathcal{D}$. The constant C in (4.22) will, in general, depend on the domain. However, as (4.22) will be used only for x-translates of certain fixed domains, that dependence will not enter. Let

$$S_{a,b} = \{(x,y) \in \Omega \mid |x-a| < b\}, \tag{4.23}$$

and suppose ζ has its support in $S_{a,b}$. Then from (4.22), if $u \in W^{2,p}_{loc} \cap H_k$,

$$\|\zeta u\|_{W^{2,p}(S_{a,b})} \leq C\left(\|\Delta(\zeta u)\|_{L^p(S_{a,b})} + \|\zeta u\|_{L^p(S_{a,b})}\right), \qquad (4.24)$$

since ζu vanishes on $\partial S_{a,b}$. The domain $S_{a,b}$ has corners, but since the support of ζu is at a positive distance from the corners the estimate can be seen to hold. Now, since $\Delta(\zeta u) = \zeta\Delta u + 2\nabla\zeta\cdot\nabla u + u\Delta\zeta$ it follows easily from (4.24) that

$$\|\zeta u\|_{W^{2,p}(S_{a,b})} \leq C'\left(\|\Delta u\|_{L^p(S_{a,b})} + \|u\|_{W^{1,p}(S_{a,b})}\right). \qquad (4.25)$$

<u>Lemma</u> 4.4. Suppose f and h satisfy condition A and let (λ,ϕ) be a solution of problem (P)(3.1) having $0 < \lambda < \mu$. Then for each integer j,

$$\|\phi\|_{C^1(S_{j,1})} \leq C\left(\|\phi\|_{W^{1,2}(S_{j,3})} + \|\phi\|^n_{W^{1,2}(S_{j,3})}\right), \qquad (4.26)$$

where C depends on μ and the parameters entering condition A.

<u>Proof</u>. The growth assumptions in AI and AII yield an estimate

$$|\Delta\phi| \leq C'(|\phi| + |\phi|^n), \qquad (4.27)$$

valid at each point (x,y), with C' depending on μ and the parameters from AI and AII. If we use $\zeta = \eta_{j-1} + \eta_j + \eta_{j+1}$ in (4.25), there obtains a $W^{2,p}$ estimate of ϕ on $S_{j,1}$ in terms of L^p estimates of $\Delta\phi$ and ϕ on $S_{j,2}$. Estimates on $S_{j,2}$ and $S_{j,3}$ are similarly related, and in each estimate the constant will involve the fixed function η. Combining these estimates with the Sobolev embedding, theorem for some $p > 2$, one has (letting $S_{j,1} = S_j$),

$$
\begin{aligned}
\|\phi\|_{C^1(S_1)} &< C_1\|\phi\|_{W^{2,p}(S_1)} < C_2\left(\|\Delta\phi\|_{L^p(S_2)} + \|\phi\|_{W^{1,p}(S_2)}\right) \\
&< C_3\left(\|\phi\|_{L^p(S_2)} + \|\phi\|^n_{L^p(S_2)} + \|\phi\|_{W^{2,2}(S_2)}\right) \\
&< C_4\left(\|\phi\|_{L^p(S_2)} + \|\phi\|^n_{L^p(S_2)} + \|\Delta\phi\|_{L^2(S_3)} + \|\phi\|_{W^{1,2}(S_3)}\right) \\
&< C_5\left(\|\phi\|_{L^p(S_2)} + \|\phi\|^n_{L^p(S_2)} + \|\phi\|_{L^2(S_3)} + \|\phi\|^n_{L^2(S_3)} + \|\phi\|_{W^{1,2}(S_3)}\right) \\
&< C\left(\|\phi\|_{W^{1,2}(S_3)} + \|\phi\|^n_{W^{1,2}(S_3)}\right),
\end{aligned}
$$

which proves (4.26).

A solution of equation (3.1) obtained in section 3 will be relevant to a physical flow only if we can restrict its size by restricting some system parameter. The previous lemma addresses this point. The following corollary continues the development in this direction. Both of these results will be used in taking limits of periodic solutions as the period grows indefinitely large.

Corollary 4.5. Let $N = R$ if $I(\phi) = R^2$ is specified as in theorem 3.2 and let

$$N = \delta = \frac{(\mu-\lambda)^{\frac{5-m}{4(m-1)}}}{\lambda\alpha-\sigma} \tag{4.28}$$

if λ is specified as in theorem 3.4. Then

$$\|\phi\|_{C^1} < C(N + N^n) \tag{4.29}$$

and, for some $\omega > 0$

$$\|\phi\|_{C^{2,\omega}} \leq C'(N + N^n)^\omega. \tag{4.30}$$

Here C and C' depend on μ and on the parameters entering condition A; in addition, C' and ω depend on N through the Hölder exponents of f and h.

<u>Proof</u>. From (3.9), (3.16) and lemmas 4.3 and 4.4, we have

$$\|\phi\|_{C^1(\Omega_k)} \leq C(N + N^n) \tag{4.31}$$

where $N = R$ or $N = \delta$. according to whether problem (PC) or (PF), respectively, is being considered. The range of ϕ is thus limited in terms of R or δ. It is assumed in condition A that both f and h are locally Hölder continuous. Hence, for the range of value of (y,t) which are relevant for the solution ϕ, $\lambda f-h$ is a member of some Hölder class, say with exponent $\omega > 0$. Regularity theory for the Laplacian (Gilbarg and Trudinger 1977) then yields (4.30).

<u>Lemma</u> 4.6. Let $L = -\Delta + q(y)$ with q continuous on $0 \leq y \leq 1$. Suppose that the smallest eigenvalue γ_1 of $-d^2/dy^2 + q$ on $0 < y < 1$ with $y(0) = y(1) = 0$ satisfies $\gamma_1 > 0$. Then for $u \in W^{2,2}(\Omega_k) \cap H_k$,

$$\|u\|_{W^{2,2}(\Omega_k)} \leq C\|Lu\|_{L^2(\Omega_k)}, \tag{4.32}$$

where C depends on γ_1.

<u>Proof</u>. From (3.7) and (4.25) it follows easily that

$$\|\eta_j u\|_{W^{2,2}(S_j)} \leq C_1\left(\|Lu\|_{L^2(S_j)} + \|u\|_{W^{1,2}(S_j)}\right), \tag{4.33}$$

where $S_j = S_{j,1}$ as in (4.23), and $j = -k, -k+1,\ldots,k-1$. It is well known

that for each $\varepsilon > 0$ there is a constant C_ε such that

$$\|u\|_{W^{1,2}(S_j)} \leq \varepsilon\|u\|_{W^{2,2}(S_j)} + C_\varepsilon\|u\|_{L^2(S_j)}, \qquad (4.34)$$

(cf. Agmon, Douglis and Nirenberg 1959, p. 698). Using the partition of unity $\{\eta_j\}$ and the fact that u is $2k$ periodic, (4.33) and (4.34) may be combined to obtain

$$\begin{aligned}
\|u\|^2_{W^{2,2}(\Omega_k)} &= \Big\|\sum_{j=-k}^{k-1} \eta_j u\Big\|^2_{W^{2,2}(\Omega)} \\
&\leq 2\sum_{j=-k}^{k-1} \|\eta_j u\|^2_{W^{2,2}(S_j)} \\
&\leq \sum_{j=-k}^{k-1} C_1\Big(\|Lu\|^2_{L^2(S_j)} + \varepsilon\|u\|^2_{W^{2,2}(S_j)} + C_\varepsilon\|u\|^2_{L^2(S_j)}\Big) \\
&\leq 2C_1\Big(\|Lu\|^2_{L^2(\Omega_k)} + \varepsilon\|u\|^2_{W^{2,2}(\Omega_k)} + C_\varepsilon\|u\|^2_{L^2(\Omega_k)}\Big) .
\end{aligned}$$

The factors of 2 appear since the rectangles S_j (and correspondingly the supports of the η_j overlap in pairs, including S_{-k} and S_{k-1} if we envision integration as taking place on the cylinder obtained from the product of $0 \leq y \leq 1$ with a circle. The choice
$\varepsilon = (4C_1)^{-1}$ yields

$$\|u\|_{W^{2,2}(\Omega_k)} \leq C_2\Big(\|Lu\|_{L^2(\Omega_k)} + \|u\|_{L^2(\Omega_k)}\Big). \qquad (4.35)$$

To absorb the term $\|u\|_{L^2}$ we use the hypothesis on γ_1 to write

$$\begin{aligned}
\gamma_1\|u\|^2_{L^2(\Omega_k)} &\leq \int\{|\nabla u|^2 + qu^2\} \\
&= \int(-\Delta u + qu)u \\
&\leq \|Lu\|_{L^2(\Omega_k)}\|u\|_{L^2(\Omega_k)}
\end{aligned}$$

obtaining $\|u\|_{L^2} < \gamma_1^{-1}\|Lu\|_{L^2}$. This last inequality together with (4.35) provides (4.32).

Next we obtain a lower bound on $\|\phi\|_{L^\infty}$ for a solution (λ,ϕ) of problem P, assuming $\lambda < \mu$. Recall that in problem (PF) we chose $\lambda < \mu$ while for problem (PC) it was shown in lemma 4.1 that $\lambda < \mu$ was a consequence of our method of obtaining a solution.

Lemma 4.7. Suppose f and h satisfy conditions AI and AII of section 1. Let $(\lambda,\phi) \in \mathbb{R} \times H_k$ with $\lambda < \mu$ and $\phi > 0$, be a solution of problem (P)(3.1). Then

$$\|\phi\|_{L^\infty(\Omega_k)} \geq r > 0,$$

where r depends on $\mu - \lambda$, but not on k.

Proof. Let $\xi > 0$ be the eigenfunction from (3.18). Since

$$\begin{aligned}\mu\int\phi f_0(y)\xi &= \int\phi(-\Delta\xi + h_0\xi)\\ &= \int(-\Delta\phi + h_0\phi)\xi\\ &= \int[\lambda f_0\phi + \lambda f_1(y,\phi) - h_1(y,\phi)]\xi,\end{aligned}$$

it follows that

$$\int \phi\xi[(\mu-\lambda)f_0(y) - \frac{\lambda f_1(y,\phi) - h_1(y,\phi)}{\phi}] = 0.$$

However, the term $\lambda f_1 - h_1$ is $o(\phi)$ as $\phi \to 0$ and $f_0\phi\xi > 0$ on Ω_k, so the integral cannot vanish if ϕ is everywhere smaller than some $r > 0$, depending on the gap $\mu - \lambda$.

Corollary 4.8. For fixed $R > 0$ in problem (PC) or for fixed λ with $\alpha^{-1}\sigma < \lambda < \mu$ in problem (PF), the solution ϕ obtained in theorem 3.2 or theorem 3.4, respectively, has nontrivial x dependence for all sufficiently large k.

Proof. In either problem we have $\|\phi\|_k < S$ and $\|\phi\|_{L^\infty(\Omega_k)} > r$ for positive constants S and r, independent of k. Suppose ϕ were independent of x. Then

$$0 < r^2 < \|\phi\|^2_{L^\infty} < C \int_0^1 \phi_y^2 < C_1 k^{-1}\|\phi\|_k < C_1 k^{-1} S$$

which is impossible for large k.

In fact, the symmetrized solutions obtained in section 3 have decay in x over $0 < x < k$, uniformly in y, as exhibited in the next lemma.

Lemma 4.9. Let (λ,ϕ), $\phi \in H_k$ be any solution of problem (P) obtained from theorems 3.2 or 3.4. Then

$$\phi(x,y) < \frac{CS(1 + S^{n-1})^{1/3}}{|x|^{1/3}}, \quad \text{if} \quad |x| < k, \tag{4.36}$$

where $S = \|\phi\|_k$.

Proof. Let

$$\beta(x) = \max_{0<y<1} \phi(x,y).$$

The equivalence of norms given by (3.9) combined with lemma 4.4 provide the bound

$$\|\phi\|_{C^1(\Omega_k)} < C(\|\phi\|_k + \|\phi\|_k^n), \tag{4.37}$$

with C independent of k. The bound (4.37) on the derivative of ϕ with respect to y implies that $\phi(x,y) > \beta/2$ on a y interval Q of length at

least $\beta/C(S+S^n)$. But $\phi = \hat{\phi}$ on Ω_k, so for y in Q and all x' satisfying $-x \le x' \le x$, $\phi(x',y) \ge \beta(x)/2$. It follows that

$$\|\phi\|^2_{L^2(\Omega_k)} \ge 2x\left(\frac{\beta}{2}\right)^2 \frac{\beta}{C(S+S^n)}.$$

Of course, $\|\phi\|^2_{L^2(\Omega_k)} \le C_1 S^2$ and so a bound for β results; viz. (4.36).

It is possible to obtain faster decay in x. Bootstrapping will furnish a higher power of x in the denominator of (4.36). However, an explicit use of the Green's function for $-\Delta + h_0 - \lambda f_0$ will yield exponential decay for ϕ, as we prove in the next lemma. For that purpose, let (γ_n, ψ_n) $n = 1,2\ldots$ be the eigenvalues and eigenfunctions of

$$-\frac{d^2\psi}{dy^2} + h_0(y)\psi - \lambda f_0(y)\psi = \gamma\psi, \tag{4.38}$$

$$\psi(0) = \psi(1) = 0,$$

with $\gamma_1 < \gamma_2 < \gamma_3 < \cdots$. It is known (Ince 1927, p. 270) that there are constants C_0 and C_1, independent of n, such that $C_0 n^2 \le \gamma_n \le C_1 n^2$ and, assuming $\gamma < \mu$, that γ_1 is at least $m_0(\mu - \gamma)$ where m_0 is the minimum of $f_0(y)$ for $0 \le y \le 1$. As a normalization assume $\int_0^1 \psi_n^2 = 1$. Since $\int_0^1 |\frac{d}{dy}\psi_n|^2 \le (\gamma_n + C_2)\int_0^1 \psi_n^2$ for some constant C_2, the Sobolev embedding inequality provides the crude estimate, $|\psi_n|_{L^\infty} \le C_3 n$ for some constant C_3 independent of n. The Green's function for $-\Delta + h_0 = \lambda f_0$ in the strip Ω with zero boundary conditions on $y = 0$ and $y = 1$ is easily computed using separation of variables (cf. Stakgold 1968, p. 163). The result is,

$$G(x-x', y, y') = \sum_n \frac{e^{-\gamma_n^{1/2}|x-x'|}}{2\gamma_n^{1/2}} \psi_n(y)\psi_n(y'), \tag{4.39}$$

and one verifies, using the crude estimate on ψ_n, that for $|x-x'| \ge b > 0$

$$|G(x-x',y\ y')| \leq Ce^{-\gamma_1^{1/2}|x-x'|} \tag{4.40}$$

with C depending on b.

Lemma 4.10. For $k > 0$ let (λ,ϕ), with $\phi \in H_k$, be a solution of problem (P)(3.1) obtained from theorem 3.2 or 3.4. Let γ_1 be the lowest eigenvalue of (4.38) and suppose that for some $q > 1$

$$|\lambda f_1(y,t) - h_1(y,t)| \leq C_0 t^q$$

for $0 \leq y \leq 1$ and $t \geq 0$. Then for $\beta < \gamma_1^{1/2}/q$,

$$\phi(x,y) \leq Ce^{-\beta|x|} \tag{4.41}$$

and

$$|\nabla\phi(x,y)| \leq C'e^{-\beta|x|} \tag{4.42}$$

for $|x| \leq k$ where C and C' depend on q, β, and $\|\phi\|_{L^\infty}$.

Remark. For the solution of problem (PC) or (PF) we have a bound on $\|\phi\|_{L^\infty}$. Further, $\mu - \lambda$ is specified or estimable and $\gamma_1 \geq m_0(\mu-\lambda)$ where $m_0 = \inf f_0$. Hence ϕ decays faster than an exponential with a rate constant proportional to $(\mu-\lambda)^{1/2}$. This dependence for the rate constant agrees with that found in the small-amplitude theories of Ter-Krikorov (1963) and Benjamin (1966).

Proof. Since ϕ is positive and decreasing on $0 \leq x \leq k$ it is enough to show that the sequence

$$a_i = \max_{0<y<1} \phi(i,y),\ i = 0,1,\ldots k,$$

satisfies

$$a_i \leq Ce^{-\beta i},\ i = 0,1,\ldots k. \tag{4.43}$$

We write the equation (P)(3.1) as

$$(-\Delta + h_0 - \lambda f_0)\phi = \lambda f_1(y,\phi) - h_1(y,\phi),$$

and abbreviate it to

$$L\phi = g(\phi), \tag{4.44}$$

suppressing the dependence on λ and y. By hypothesis,

$$|g(\phi)| \leq c_0\phi^q.$$

Consider the cutoff function η_j introduced earlier in this section, but now extend it to be 2k-periodic in x. Maintaining the notation η_j for the new function we write equation (4.44) as

$$L\phi = \eta_j g(\phi) + (1-\eta_j)g(\phi). \tag{4.45}$$

The operator L is invertible in H_k and hence we can express ϕ as $\phi = \theta + \chi$ where

$$L\theta = \eta_j g(\phi)$$

and

$$L\chi = (1-\eta_j)g(\phi).$$

From the growth estimate on g, lemma 4.6, and the Sobolev embedding theorem one concludes that

$$|\theta(j,y)| \leq c_2 a_{j-1}^q, \tag{4.46}$$

for $j = 1,2,\ldots k$. For χ one has the estimate

$$|\chi(j,y)| \leq c \int_\Omega |G(j-x',y\ y')|\ (1-\eta_j(x'))\phi^q(x',y'). \tag{4.47}$$

Since for each y, ϕ is even in x, 2k-periodic, and decreasing on the interval $0 \leq x \leq k$, it follows that for $0 \leq j \leq k$ and $0 \leq x' \leq j$,

$$\phi(j+x',y) \leq \phi(j-x',y). \tag{4.48}$$

Since G is even in its first variable, it follows from (4.48) that the contribution to the integral in (4.47) taken over $0 \leq x' \leq j$ is at least as large as the contribution from integration over $j \leq x' \leq 2j$. Using this last observation and the inequality (4.40) we conclude that

$$|\chi(j,y)| \le C_1\left(\int_0^j dx' + \int_j^{2j} dx' + \int_{|x'-j|\ge j} dx'\right)\int_0^1 (1-\eta)|G|\phi^q dy'$$

$$\le C_1\left(2\int_0^j dx' + \int_{|x'-j|\ge j} dx'\right)\int_0^1 (1-\eta)|G|\phi^q dy'$$

$$\le 2C_1 \sum_{i=0}^{j-1} \int_i^{i+1} dx' \int_0^1 (1-\eta)|G(j-x',y,y')|\phi^q dy' + \tag{4.49}$$

$$+ C_1' \int_{|x'-j|\ge j} dx' \int_0^1 |G(j-x',y,y')|\phi^q dy'$$

$$\le C_3 \left\{ \sum_{i=0}^{j-1} e^{-\gamma_1^{1/2}(j-i)} a_i^q + \frac{e^{-\gamma_1^{1/2} j}}{\gamma_1^{1/2}} \|\phi\|_{L^\infty}^q \right\}.$$

Here we have made essential use of the vanishing of $(1-\eta(x'))G(j-x',y,y')$ when $|x'-j| < 1/4$ so as to estimate it by a suitable constant times $e^{-\gamma_1^{1/2}}$ for $j-1 \le x' \le j$. Combining (4.46) and (4.49) to estimate $\phi = \theta + \chi$ we find that

$$a_j \le C_2 a_{j-1}^q + C_3 \left\{ \sum_{i=0}^{j-1} e^{-\gamma_1^{1/2}(j-i)} a_i^q + \frac{e^{-\gamma_1^{1/2} j}}{\gamma_1^{1/2}} \|\phi\|_{L^\infty}^q \right\} \tag{4.50}$$

for $j = 1,2,\ldots,k$. Or, if a_0 is redefined to be

$$a_0 = \sup_{0\le y\le 1} \left\{ \phi^q(0,y) + \frac{\|\phi\|_{L^\infty}^q}{\gamma_1^{1/2}} \right\}^{1/q}$$

then

$$a_j \le C_2\, a_{j-1}^q + C_3 \sum_{i=0}^{j-1} e^{-\gamma_1^{1/2}(j-i)} a_i^q \tag{4.51}$$

for $j = 1,2,\ldots,k$.

Given β, $0 < \beta < \frac{1}{q}\gamma_1^{1/2}$ and $B > 0$ we can choose a positive integer N depending on β and B so that for $i = 0,1,2,\ldots,N$,

$$a_i < Be^{\beta(N-i)}. \tag{4.52}$$

To see that there is such an N we first observe that since ϕ is bounded and, by lemma 4.9, decays like $x^{-1/3}$,

$$a_i < \frac{E}{(1+i)^{1/3}},$$

for $i = 0,1,\ldots k$, where E is a positive constant depending on $\|\phi\|_{L^\infty}$. The function $e^{\beta i}/(1+i)^{1/3}$ is convex in i and thus its maximum on $0 < i < N$ occurs at $i = 0$ or $i = N$. Hence (4.52) will hold provided N is chosen so that

$$\max(1,\ e^{\beta N}/(1+N)^{1/3}) < \frac{B}{E}\, e^{\beta N}.$$

We now proceed inductively, supposing that for some $j > N$ (4.52) holds for $i = 0,1,\ldots,j-1$. From (4.51),

$$\begin{aligned} a_j &< C_2B^q e^{q\beta(N-j+1)} + C_3 \sum_{i=0}^{j-1} e^{-\gamma_1^{1/2}(j-i)} B^q e^{q\beta(N-i)} \\ &= Be^{\beta(N-j)}\Big(C_2B^{q-1}e^{(q-1)\beta(N-j)+q\beta} + C_3B^{q-1}e^{(q-1)\beta(N-j)} \sum_{i=0}^{j-1} e^{(q\beta-\gamma_1^{1/2})(j-i)}\Big) \\ &< Be^{\beta(N-j)}[C_2B^{q-1}e^{\beta} + C_3B^{q-1}e^{\beta-\gamma_1^{1/2}}(1 - e^{q\beta-\gamma_1^{1/2}})^{-1}]. \end{aligned} \tag{4.53}$$

If B is chosen so that the contents of the square bracket in (4.53) add to 1, then the desired inequality (4.52) is valid for $i = j$ and induction shows it to be valid for $i = 0,1,\ldots,k$. Thus (4.43) and (4.41) hold with $C = Be^{\beta N}$.

The decay (4.42) for $\nabla\phi$ follows readily from (4.41) in view of (4.22), (4.27), and the Sobolev embedding theorem.

5. THE EXISTENCE OF SOLITARY WAVES

In this section we show the existence of solitary-wave solutions of problems (PC)3.3) and (PF)(3.4). These solutions will be obtained from the foregoing periodic wave trains in the limit as the period becomes unbounded.

Theorem 5.1. Suppose f and h satisfy condition A of section 3 and let α, σ, m be as defined therein; let μ be the lowest eigenvalue of (3.17). Suppose further that $m < 5$, $2\alpha^{-1}\sigma < \mu$ and $|\lambda f(y,t) - h(y,t)| \leq Ct^q$ for some $q > 1$.

I. Let $R > 0$ be given. For each $k > 0$, let (λ_k, ϕ_k) be a solution of problem (PC) given by theorem 3.2 with $I(\phi_k) = R^2$, where $I(\phi)$ is given by (3.14). Then there is an increasing sequence of half-periods $k(j) \to \infty$, $j = 1,2,\cdots$, and a solution (λ, ϕ) of (P) satisfying

1) $\phi > 0$ and $\phi = \hat{\phi}$ on Ω,

2) $I(\phi) = R^2$,

3) $\|\phi\|_{C^{2,\omega}} \leq C(R + R^n)^\omega$ (cf. 4.30),

4) $|\phi| \leq Ce^{-\beta|x|}$; $|\nabla\phi| \leq Ce^{-\beta|x|}$ (cf 4.41, 4.42),

5) as $j \to \infty$, $\lambda_{k(j)} \to \lambda$, $0 < \lambda < \mu$, and

6) as $j \to \infty$, $\phi_{k(j)}$ converges to ϕ,

uniformly in C^2 on bounded subsets of Ω.

II. Let λ in $(\alpha^{-1}\sigma, \mu)$ be given. For each $k > 0$ let (λ, ϕ_k) be a solution of (PF) given by theorem 3.4. Then there is a sequence $k(j) \to \infty$, $j = 1,2,\cdots$, as in part I and a solution (λ, ϕ) of (P) satisfying properties 1), 4) and 6) of part I as well as

$$2') \quad \|\phi\|_{W^{1,2}(\Omega)} \leq C\{\lim_{j\to\infty} \|\phi\|_{k(j)}\|_{k(j)}\} < \infty,$$

and

$$3') \quad \|\phi\|_{C^{2,\omega}} \leq C(\delta + \delta^n)^\omega,$$

where δ is given by (4.30).

Proof. We prove part I. The proof of part II is almost identical. Consider the sequence ϕ_k, $k = 1,2,\ldots,$ restricted to Ω_1. The estimate (4.30) for some $\omega > 0$ and the Arzela-Ascoli lemma imply that a subsequence $\phi_{11}, \phi_{12}, \phi_{13}, \cdots,$ converges in $C^2(\Omega_1)$ to a function $\phi^{(1)}$. Similarly $\phi_{11}, \phi_{12}, \cdots,$ restricted to Ω_2 contains a subsequence $\phi_{21}, \phi_{22}, \cdots,$ converging in $C^2(\Omega_2)$ to a function $\phi^{(2)}$ which, of course, coincides with $\phi^{(1)}$ on Ω_1. Continuing the process and extracting the diagonal sequence ϕ_{kk} we obtain a sequence which we denote $\phi_{k(j)}$, $j = 1,2,\cdots$. Letting ϕ be the element of $C^2(\Omega)$ defined to be $\phi^{(n)}$ on Ω_n, it is clear that 6) holds. Without loss of generality we can assume that $k(j)$ is a sequence of positive integers for which $k(j) \geq j$ and $\lambda_{k(j)}$ converges to a value λ. Since

$$\mu(1 - C(R)) \leq \lambda_{k(j)} \leq \mu(1 - \eta(R))$$

according to lemmas 4.1 and 4.2, λ has the same bounds, so assertion 5) holds. Moreover, (λ,ϕ) is a classical solution of the problem (P),

$$\left.\begin{aligned} -\Delta\phi + h(y,\phi) &= \lambda f(y,\phi), \\ \phi(x,0) = \phi(x,1) &= 0, \end{aligned}\right\}$$

since each pair $(\lambda_{k(j)}, \phi_{k(j)})$ is a solution for $j = 1,2,\ldots$ and all terms in the equation converge uniformly on bounded sets.

Since $\phi_{k(j)}$ and $\nabla\phi_{k(j)}$ converge uniformly on bounded sets, property 4) follows immediately from lemma 4.10. Further $\phi = \hat{\phi}$ on Ω since, for each j, $\phi_{k(j)}$ is even in x, nonnegative, and nonincreasing on

$0 < x < k(j)$. The estimate $\|\phi\|_{L^\infty} > r > 0$ also follows from the uniform convergence on Ω_1 in the light of lemma 4.7. So $\phi > 0$ follows exactly as in the proof of theorem 3.2. Assertion 2) derives from the convergence of $\phi_{k(j)}$ in C^1, uniformly on bounded sets, together with inequality 4). Finally, the Hölder estimates from lemma 4.5 are preserved in the limit, yielding 3) and completing the proof.

6. EXAMPLES

The application of the foregoing theory is exemplified in the present section for a specific class of stratified fluid flows. The coordinate system and notation remain as set forth in section 2. It is assumed that, in the undisturbed state of the system, the density ρ is given by,

$$\rho(y) = \rho_0(1 - ay)^r, \quad \text{for} \quad 0 \leqslant y \leqslant 1, \tag{6.1}$$

where ρ_0 and r are positive, and $0 < a < 1$. This family of density profiles is supplemented with a uniform primary velocity,

$$U(y) = c_p, \quad \text{for} \quad 0 \leqslant y \leqslant 1.$$

We search for waves of permanent form whose velocity of propagation downstream (in the direction of increasing x) is $\bar{c}$. The crests of the waves are brought to rest by considering the system in a frame of reference moving to the right at speed $\bar{c}$. In the moving frame of reference, the primary fluid velocity is

$$W(y) = U(y) - \bar{c} = c_p - \bar{c} = c, \tag{6.2}$$

say. From the specifications (6.1) and (6.2), the functions denoted by f and h in earlier sections may be determined explicitly in a straightforward manner (cf. Benjamin 1971, §6.1) which is now outlined. The outcome of this calculation is the guarantee that, subject to some restriction on the parameters a and r, the system that has the primary specifications (6.1) and (6.2) can support both solitary waves and periodic waves of permanent form, the latter being the analogue of the classical cnoidal waves.

To begin the computation of f and h, note that the pseudo-stream-function associated to the primary flow is

$$\Psi(y) = \int_0^y \rho^{1/2}(z)W(z)dz$$

$$= \frac{c\,\rho_0^{1/2}}{a[1+(r/2)]}\,[1-(1-ay)^{1+(r/2)}].$$

It will be convenient to let $s = 1 + (r/2)$ and $\gamma = c\,\rho_0^{1/2}/as$. With this notation, we have

$$\Psi(y) = \gamma[1-(1-ay)^s]. \tag{6.3}$$

Note, for later use, that

$$\Psi_{yy}(y) = -s(s-1)a^2\gamma(1-ay)^{s-2}. \tag{6.4}$$

As all variables need eventually to be cast in terms of the pseudo-stream-function, it is convenient to express y as a function of Ψ. The expression is,

$$y = Y(\Psi) = \frac{1}{a}\,[1-(1-\frac{1}{\gamma}\,\Psi)^{1/s}]. \tag{6.5}$$

Then ρ may be expressed in terms of Ψ as

$$\rho(\Psi) = \rho_0(1-\frac{1}{\gamma}\,\Psi)^{r/s}. \tag{6.6}$$

In particular,

$$\rho'(\Psi) = -\frac{\rho_0 r}{\gamma s}\,(1-\frac{1}{\gamma}\,\Psi)^{(s-2)/s}. \tag{6.7}$$

Similarly, combining (6.4) and (6.5), it follows that

$$\Psi_{yy} = -s(s-1)a^2\gamma(1-\frac{1}{\gamma}\,\Psi)^{(s-2)/s}. \tag{6.8}$$

Remember the dynamical condition satisfied by the flow, that the total head,

$$H = p + gy\rho + \frac{1}{2}\,\Psi_y^2, \tag{6.9}$$

is conserved along each pseudo-stream-line. Thus H is a function of Ψ alone, as pointed out already in section 2. Differentiating (6.9) with respect to Ψ, and using the hydrostatic law,

$$p_y = -g\rho,$$

there appears

$$H'(\Psi) = gy\rho'(\Psi) + \Psi_{yy}, \tag{6.10}$$

which is (2.5) in the special case of a parallel flow. This completes the calculations regarding the primary flow.

Let ψ denote the pseudo-stream-function for the perturbed flow, and write

$$\psi(x,y) = \Psi(y) + j\phi(x,y), \tag{6.11}$$

where j is a constant, to be fixed presently. (It may be either positve or negative, corresponding to waves of depression or elevation, respectively.) According to (2.5), ψ satisfies the equation

$$-\Delta\psi + H'(\psi) = gy\rho'(\psi). \tag{6.12}$$

Now, (6.10), (6.11) and (6.12) combine to yield the equation,

$$-j\Delta\phi + \Psi_{yy}(\Psi + j\phi) - \Psi_{yy}(\Psi) = -g\rho'(\Psi + j\phi)\{Y(\Psi + j\phi) - Y(\Psi)\},$$

satisfied by ϕ (cf. Benjamin 1971, eqn. 6.10). Making use of (6.7) and (6.8), this may be worked out completely as

$$-\Delta\phi + h(y,\phi) = \lambda f(y,\phi), \tag{6.13}$$

where $\lambda = |g\ \rho_0 r/as^2 j\gamma|$,

$$\left.\begin{aligned} f(y,\phi) &= s(1 - ay)^{s-1}\mathrm{sgn}(j)\bar{f}\Big(\frac{j\phi}{\gamma(1-ay)^s}\Big), \\ \bar{f}(z) &= (1 - z)^{(s-2)/s}[1 - (1-z)^{1/s}], \end{aligned}\right\} \tag{6.14}$$

and

$$\left.\begin{aligned} h(y,\phi) &= \frac{s(s-1)\gamma a^2}{j}(1 - ay)^{s-2}\,\bar{h}\Big(\frac{j\phi}{\gamma(1-ay)^s}\Big), \\ \bar{h}(z) &= [1 - (1 - z)^{(s-2)/s}]. \end{aligned}\right\} \tag{6.15}$$

The selection $|j| = \gamma$ simplifies the foregoing formulae and renders ϕ dimensionless. If $j = \gamma$, the formulae provide waves of depression, in which the streamlines are displaced from their undisturbed configuration toward the lower boundary, whilst $j = -\gamma$ corresponds to waves of elevation, in which the streamlines are displaced upward from their undisturbed postion. (If $j = \gamma$, then $\rho^{1/2}v = -\psi_x = -\gamma\phi_x$, where v is the vertical velocity of the flow. Considering the shape of ϕ, as established by our theory, it is concluded that the vertical velocity is negative to the left of the wave's crest, corresponding to a wave of depression. The opposite conclusion obviously holds if $j = -\gamma$.) For the sample primary flow described by (6.1) and (6.2), these two distinct cases correspond to different ranges of the parameter r, and are considered separately below. Note that if $|j| = \gamma$, then

$$\lambda = \frac{gra}{c^2},$$

and so is proportional to the inverse-square of the velocity of the permanent-wave in question, relative to the velocity of the primary flow.

It is worth remembering that equation (6.12) has, as an essential ingredient in its derivation, the presumption that all the pseudo-stream-lines begin at $-\infty$ and terminate at $+\infty$. It follows that ψ must take its values in the range $[0, \Psi(1)]$. Since ϕ vanishes at $y = 0$ and $y = 1$, these requirements are implied by the condition

$$\psi_y = \Psi_y + j\phi_y > 0, \quad \text{for} \quad (x,y) \in R \times (0,1). \tag{6.16}$$

In the cases of special interest here, this amounts to,

$$\phi_y(x,y) > -sa(1 - ay)^{s-1}, \tag{6.17}$$

in case $j = \gamma$, and

$$\phi_y < sa(1 - ay)^{s-1}, \tag{6.18}$$

for $j = -\gamma$.

Waves of Depression

Here we concentrate on the case $j = \gamma$. According to the preceding discussion, the total head H and the density ρ are defined only for values of ψ lying in the range $[0, \Psi(1)]$. (In this case, $\Psi(1) = \gamma[1 - (1 - a)^s]$.) Hence the functions f and h are necessarily as displayed in (6.14) and (6.15) for values of y in $[0,1]$ and $t \geq 0$ such that $\Psi(y) + \gamma t$ lies in the interval $[0, \Psi(1)]$. This implies exactly that the pair (y,t) must respect the inequality,

$$0 \leq t \leq (1 - ay)^s - (1 - a)^s = \tau(y), \qquad (6.19)$$

say. Note that $0 \leq \tau(y) \leq \tau(0) = \Psi(1)/\gamma < 1$, for all y in $[0,1]$. The definition of f and h for values of (y,t) outside the domain defined in (6.19) is entirely at our disposal. It is technically convenient, as in section 3, to require both f and h to be odd functions of the variable t. So, only the extension of the domain of f and h to encompass all $t \geq 0$ need be considered. For the present, we content outselves with extending f and h to the set

$$\Gamma = \{(y,t) : 0 \leq y \leq 1 \text{ and } 0 \leq t \leq \tau(0)\} \qquad (6.20)$$

simply by using the formulas in (6.14) and (6.15). Since $\tau(0) < 1$, this raises no obvious difficulties. The definition of f and h for $t > \tau(0)$ will be considered subsequently. Of course, as discussed before, it is required to assure, in due course, that the permanent-wave solutions obtained satisfy (6.17) and so do not depend on the particular extension of f and h that is chosen.

First, f and h are shown to satisfy conditions, AI, AII and AIII, at least when restricted to the set Γ. For this, we suppose

$$a < \pi/[\pi + ((s-1)(2-s))^{1/2}]. \qquad (6.21)$$

The restriction on a is a technical one and not a real hindrance. As r varies between 0 and 2, s varies over the interval $[1,2]$. Consequently, the constraint on a asks no more than $a < .85$. Practical values of a are typically very much smaller. The Brunt-Väisälä frequencies $N = (ag)^{1/2}$ measured in the field are of the order .01 to .02 corresponding to an a of order 10^{-7}/cm. (Imberger, Thompson and Fandry 1976). In the laboratory, Mahony and Pritchard (1981) measured $N \approx 1$ or $a \approx .003$/cm for a salt stratification near the limit of salt solubility.

It will be convenient to represent $\bar{f}$ and $\bar{h}$ as power series in z and for that purpose to use

$$(1-z)^{-\nu} = 1 + \nu z + \nu(\nu+1)\frac{z^2}{2!} + \nu(\nu+1)(\nu+2)\frac{z^3}{3!} + \cdots .$$

Then, with

$$P_k(\xi) = (\xi-1)\xi(\xi+1)\cdots(\xi+k-2), \tag{6.22}$$

the series can be written

$$\begin{aligned}\bar{f}(z) &= (1-z)^{-(\frac{2}{s}-1)} - (1-z)^{-(\frac{1}{s}-1)} \\ &= \sum_{k=1}^{\infty} [P_k(\tfrac{2}{s}) - P_k(\tfrac{1}{s})]\frac{z^k}{k!}\end{aligned} \tag{6.23}$$

and

$$\bar{h}(z) = -\sum_{k=1}^{\infty} P_k(\tfrac{2}{s})\frac{z^k}{k!}, \tag{6.24}$$

convergent for $|z| < 1$.

It is easily seen that, in reference to hypothesis AI of section 3, $f_0(y) = \partial_t f(y,0)$ is given by

$$f_0(y) = \frac{1}{(1-ay)}. \tag{6.25}$$

It is strictly positive and infinitely differentiable for y in $[0,1]$ and so is certainly locally Hölder continuous in that range. In consequence of

(6.14) and (6.23),

$$f_1(y,t) = f(y,t) - f_0(y)t$$

$$= s(1-ay)^{s-1}\sum_{k=2}^{\infty} [P_k(\tfrac{2}{s}) - P_k(\tfrac{1}{s})] \frac{1}{k!} [\frac{t}{(1-ay)^s}]^k$$

$$= (1-ay)^{s-1}\{\frac{3-s}{s} \frac{1}{2!} [\frac{t}{(1-ay)^s}]^2 + \frac{7-s^2}{s^2} \frac{1}{3!} [\frac{t}{(1-ay)^s}]^3 + \ldots\} \tag{6.26}$$

$$= \sum_{k=2}^{\infty} a_k(y)t^k.$$

Since $0 \leq t \leq \tau(0) < 1$ it is easily seen that f_1 is locally Hölder continuous in Γ and that

$$f_1(y,t) \leq D(1 + t^2) \tag{6.27}$$

in Γ, for a suitably large D. Since $P_k(\frac{2}{s}) \geq 0$ and $P_k(\frac{1}{s}) \leq 0$ for $1 \leq s \leq 2$ and $k = 1,2,3,\ldots,$ it follows that $a_k(y) \geq 0$ for all k. Thus condition AI is seen to hold with

$$m = n = 2,\ \alpha = \frac{3-s}{2s},\ \text{and}\ d = D. \tag{6.28}$$

Regarding condition AII relative to h, it is easily seen from (6.15) and (6.24) that

$$h_0(y) = \frac{(s-1)(s-2)a^2}{(1-ay)^2} \tag{6.29}$$

and that

$$h_1(y,t) = h(y,t) - h_0(y)t$$

$$= -\frac{s(s-1)a^2}{(1-ay)^{2-s}} \sum_{k=2}^{\infty} \frac{P_k(\frac{2}{s})}{k!} [\frac{t}{(1-ay)^s}]^k$$

$$= -\frac{s(s-1)a^2}{(1-ay)^{s+2}} \{\frac{(2-s)2}{s^2} \frac{t^2}{2!} + \frac{(2-s)2(2+s)}{s^3} \frac{t^3}{3!(1-ay)^s} + \ldots\} \tag{6.30}$$

$$= -\sum_{k=2}^{\infty} b_k(y)t^k,$$

where $b_k \geq 0$ for all k. Clearly h_0 and h_1 are locally Hölder continuous in Γ. Since $b_k \geq 0$, (6.30) implies that

$$\sigma' t^2 \leq h_1(y,t) \leq \sigma t^2 \qquad (6.31)$$

in Γ, where $\sigma = 0$ and σ' is a suitably chosen constant. By choosing a possibly larger value of D in (6.27), the analogous estimate for $|h_1(y,t)|$ can be assured. Lastly,

$$\frac{\partial h}{\partial t} = \frac{(s-1)(s-2)a^2}{(1-ay)^2}\left[1 - \frac{t}{(1-ay)^s}\right]^{-2/s}, \qquad (6.32)$$

so the condition $\frac{\partial h}{\partial t} > -\pi^2$ amounts to the restriction that the quantity on the right-hand side of (6.32) exceed $-\pi^2$ provided $(y,t) \in \Gamma$. In view of (6.19), the latter is implied if, for $0 \leq y \leq 1$,

$$\frac{(s-1)(2-s)a^2}{(1-ay)^2}\{1 - [1 - (\frac{1-a}{1-ay})^s]\}^{-2/s} < \pi^2.$$

A short computation shows this restriction to be equivalent to the supposition (6.21) that

$$a < \frac{\pi}{\pi + [(s-1)(2-s)]^{1/2}}.$$

Thus the condition AII is seen to hold on Γ.

Finally, the hypothesis AIII is addressed. Recall the definitions of F_1 and H_1, that

$$F_1(y,t) = 2\int_0^t f_1(y,p)dp \quad \text{and} \quad H_1(y,t) = 2\int_0^t h_1(y,p)dp.$$

Since $\sigma = 0$ in AII, $\sigma\alpha^{-1} = 0$. Hence it is to be verified that for any $\lambda > 0$, there is a θ in $(0,1)$ such that

$$\lambda F_1(y,t) - H_1(y,t) \leq \theta[\lambda f_1(y,t) - h_1(y,t)]t. \qquad (6.33)$$

Rather than establish (6.33) in Γ we will show that f_1 and h_1 satisfy a slightly stronger condition, which, in view of the following lemma, will guarantee that for suitable extensions of f_1 and h_1 outside Γ, (6.33)

will hold for all $t \geq 0$. The extensions will be such that AI and AII are satisfied as well.

<u>Lemma</u> 6.1. Let g be a nonnegative C^1 function defined on $0 \leq t \leq t_0$ for some $t_0 > 0$. Suppose there is a constant θ with $2/3 \leq \theta < 1$ such that

$$2g(t) \leq \theta(g'(t)t + g(t)) \tag{6.34}$$

on $[0, t_0]$. Then it is possible to extend g to a C^1 function $\tilde{g}$ defined for all $t \geq 0$ and satisfying

$$\tilde{G}(t) \leq \theta\tilde{g}(t)t \tag{6.35}$$

where

$$\tilde{G}(t) = 2\int_0^t \tilde{g}(p)dp. \tag{6.36}$$

<u>Proof</u>. If (6.34) is integrated from 0 to t, it yields

$$G(t) \leq \theta g(t)t, \tag{6.37}$$

for $0 \leq t \leq t_0$, with G defined in analogy with $\tilde{G}$. In fact all one need assume in the proof is that the inequality (6.37) holds for $0 \leq t \leq t_0$ and that (6.34) is valid at the endpoint $t = t_0$. Let

$$\tilde{g}(t) = g(t_0) + g'(t_0)(t-t_0) + q(t-t_0)^2, \tag{6.38}$$

with q to be chosen. We need check (6.35) only for $t > t_0$, given that (6.37) holds. Since

$$\tilde{G}(t) = G(t_0) + 2g(t_0)(t-t_0) + g'(t_0)(t-t_0)^2 + \frac{2}{3}q(t-t_0)^3$$

and

$$\theta\tilde{g}(t)t = \theta[g(t_0) + g'(t_0)(t-t_0) + q(t-t_0)^2](t-t_0)$$

$$+ \theta[g(t_0) + g'(t_0)(t-t_0) + q(t-t_0)^2]t_0,$$

the desired inequality takes the form,

$$G(t_0) - \theta g(t_0)t_0 \leq (t-t_0)\,[(\theta-2)g(t_0) + \theta g'(t_0)t_0]$$
$$+ (t-t_0)^2[(\theta-1)g'(t_0) + \theta q t_0]$$
$$+ (t-t_0)^3[(\theta - \tfrac{2}{3})q].$$

In view of our assumptions, the inequalities will hold if

$q = (1-\theta)g'(t_0)/\theta t_0$, thus completing the proof.

For each y the function $g(t) = \lambda f_1(y,t) - h_1(y,t)$, defined for $0 < t < t_0 = \tau(0)$, is a candidate to which we may attempt to apply lemma 6.1. If we show g satisfies (6.34), then it can be extended to $t > t_0$ using (6.38) so that (6.33) is satisfied. In order to maintain the condition $\frac{\partial h}{\partial t} > -\pi^2$, which is satisfied on $0 < t < \tau(0)$, extend h_1 linearly and add the quadratic term to f_1.

That is, for $t > t_0$, define

$$\begin{aligned} f_1(t) &= f_1(t_0) + f_1'(t_0)(t-t_0) + \lambda^{-1}q(t-t_0)^2 \\ h_1(t) &= h_1(t_0) + h_1'(t_0)(t-t_0), \end{aligned} \tag{6.39}$$

where the y dependence has been suppressed throughout. One can choose q larger, if need be, to guarantee that AI continues to hold for the extended f_1.

The inequality (6.34), with $\theta = 2/3$, holds for any power t^k provided $k \geq 2$. Since (6.34) is linear in g, it holds for

$$\lambda f_1 - h = \sum_{k=2}^{\infty} (\lambda a_k + b_k)t^k,$$

a series with nonnegative coefficients.

It has been verified that the functions f and h arising from the density profile $\rho_0(1-ay)^r$ for $0 < r \leq 2$ satisfy condition A. According to theorems 3.2 and 3.4 there are x-periodic solutions (λ,ϕ) of equation(3.1)

obtainable by specifying R in problem (PC)(3.3) or λ in problem (PF)(3.4). According to corollary 4.5, condition (6.17) will be satisfied provided we restrict R to a range $0 < R < R_0$ or λ to a range $\lambda_0 < \lambda < \mu$. With these restrictions in force,

$$\psi(x,y) = \frac{c\,\rho_0^{1/2}}{a[1-(r/2)]}\,[1-(1-ay)^{1+(r/2)} + \phi(x,y)],$$

with (6.40)

$$c = \left(\frac{gra}{\lambda}\right)^{1/2},$$

has range in $[0, \Psi(1)]$; is a solution of Yih's equation (2.5); and provides, for each $k > 0$, a train of depression waves, $2k$-periodic in x. Since $m < 5$ and $0 = 2\alpha^{-1}\sigma < \mu$, lemma 4.1 implies that $\lambda < \mu$ when R is specified. Corollary 4.8 then guarantees that the x-dependence is nontrivial for all sufficient large k.

For the same ranges of R or λ there are solitary-wave solutions of Yih's equation. These are still described by (6.40), the pair (λ,ϕ) now being given by theorem 5.1. Since $|\lambda f_1 - h_1| \leq C_0 t^2$, the deviation of the velocity fields of these solitary waves from the velocity field of the trivial flow decay exponentially as $|x| \to \infty$.

For r in the range $0 < r \leq 2$ (or $1 < s \leq 2$), the nonlinear functions f and h arising from $\rho = \rho_0(1-ay)^r$ satisfy condition A on a domain Γ which arose naturally in our analysis, and the applicability of the results of sections 3-5 was limited only by having to insure that $0 \leq \psi \leq \Psi(1)$. For $r > 2$ we no longer deal exclusively with power series of positive terms in expanding f and $-h$ and must therefore impose further restrictions on the range of $\phi(x,y)$ to apply the general theory. For $2 < r < 4$ we continue to use $j = \gamma$ so f and h are described by (6.26) and

(6.30) respectively. Naturally, f_0 and h_0 will not change. To simplify matters, and to exhibit certain features of the regime in which $2 < s < 3$, attention is given to the situation in which $\theta = 3/4$ and $0 < a \leq .2$, so that $(1-a)^s > 1/2$. Since

$$-\frac{1}{4}(k-1)! \leq P_k(\xi) \leq 0, \tag{6.41}$$

for $0 \leq \xi \leq 1$, and since $\frac{1}{1-ay} \geq 1$, it follows that

$$f_1(y,t) \geq \frac{3-s}{2s}t^2 - s\sum_{k=3}^{\infty}\frac{(2t)^k}{4k}. \tag{6.42}$$

From (6.42) it is easy to see that the choice

$$\alpha = \frac{3-s}{16s} \tag{6.43}$$

more than suffices to guarantee $f_1 \geq \alpha t^2$ on the set

$$\Gamma = \{(y,t) \mid 0 \leq y \leq 1;\ 0 \leq t \leq \frac{3-s}{32s}\} \tag{6.44}$$

(the choices here are influenced by condition AIII). The remainder of AI is verified as before. Again using (6.41), we see that h_1, which is in this range given by a power series with non-negative coefficients, satisfies

$$0 \leq h_1 \leq \frac{s(s-1)a^2}{(1-a)^{s+2}}\left[\frac{s-2}{s^2}t^2 + \frac{s-2}{s}t^2\sum_{k=3}^{\infty}\frac{(2t)^{k-3}}{k}\right],$$

and hence for $(y,t) \in \Gamma$,

$$\sigma' t^2 \leq h_1(y,t) \leq \sigma t^2,$$

with $\sigma' = 0$ and

$$\sigma = \frac{4(s-1)(s-2)a^2}{s}. \tag{6.45}$$

Since $\partial h/\partial t \geq 0$ the remainder of AII is plainly satisfied.

If condition (6.34) is valid in Γ, for $g = \lambda f_1 - h_1$, then appeal to lemma 6.1 assures that f_1 and h_1 may be extended to all of $[0,1] \times \mathbf{R}$ whilst still satisfying condition AIII. As before, the extensions described in (6.39) will serve to maintain conditions AI and AII. Inequality

(6.34) takes the form,

$$\sum_{k=2}^{\infty} [(k+1)\theta-2](\lambda a_k(y) - b_k(y)]t^k > 0, \tag{6.46}$$

using the notation of (6.26) and (6.30). The condition (6.46) will follow from the more explicit inequality,

$$\begin{aligned} &\lambda\{(3\theta-2)\frac{3-s}{2s}t^2 - s\sum_{k=3}^{\infty}\frac{[(k+1)\theta-2]}{4k(1-a)^{(k-2)s}}t^k\} \\ &\quad + a^2\{(3\theta-2)\frac{(s-1)(s-2)}{s(1-a)}t^2 + \frac{2-s}{s}\sum_{k=3}^{\infty}\frac{[(k+1)\theta-2]}{k(1-a)^{(k-2)s+1}}t^k\} > 0, \end{aligned} \tag{6.47}$$

obtained using (6.41) and the restriction $2 < s < 3$. With $\theta = 3/4$, $(k+1)\theta-2 < 3k/4$ and the expression multiplying λ in (6.47) is at least

$$t^2 [\frac{1}{4}\frac{3-s}{2s} - (\frac{3}{4})^2 \sum_{k=3}^{\infty} (2t)^{k-2}]$$

and so more than $t^2(3-s)/16s$ for (y,t) in Γ. Hence (6.47) will follow if

$$t^2[\lambda \frac{3-s}{16s} - \frac{a^2(s-1)(s-2)}{s(1-a)} (1 + \sum_{k=3}^{\infty} \frac{3}{4} (2t)^{k-2})] > 0. \tag{6.48}$$

In AIII we assume that

$$\lambda > \frac{\sigma}{\alpha} = \frac{64(s-1)(s-2)a^2}{3-s}, \tag{6.49}$$

and because of this, a simple estimate shows (6.47) to be true in Γ when (6.48) holds. To apply lemma 4.1 we require $\sigma/\alpha < \mu/2$; i.e.

$$\frac{64(s-1)(s-2)a^2}{3-s} < \frac{\mu}{2}. \tag{6.50}$$

This last condition also guarantees that the interval $(\sigma/\alpha, \mu)$ is nonempty (cf. theorem 5.1). With the extra condition (6.49) fulfilled one can, as in the case $0 < r \leq 2$ assert that for $2 < r < 4$ $(2 < s < 3)$ in (6.1) there

are periodic and solitary waves of depression for a suitable range of "energies" R or supercritical velocities c, derivable from (6.40).

In verifying the condition A in the range $2 < r < 4$ $(2 < s < 3)$ we forfeited any advantage of having s near 2 to illustrate that the range of allowable amplitudes vanishes as s approaches 3. This is a manifestation of a transition which takes place, from waves of depression to waves of elevation. In the limit of vanishing stratification $(a \to 0)$ the transition occurs at $r = 4$. For positive a the transition occurs at an $r > 4$. This can be established by relaxing condition A and using the techniques of section 3. For exammple, in problem (PF) one need put conditions only on the combination $\lambda f - h$. As noted earlier, condition A is a compromise aimed at limiting the length of the exposition.

Waves of Elevation

Here we are concerned with the profile (6.1) with an exponent $r > 4$, corresponding to $s > 3$. The terms f_0 and h_0 are unaffected by the choice of j in (6.14) and (6.15), but an inspection of (6.26) shows that the term quadratic in t in $-f_1(y,-t)$ has a positive coefficient for $s > 3$, dictating the choice $j = -\gamma$. To simplify writing, and so that we can make use of the foregoing formulas, we use a superscript $+$ to designate functions associated with the range $s > 3$. For example, $f_1^+(y,t) = -f_1(y,-t)$.

The range restriction on t now derives from

$$0 \leq \Psi - \gamma t \leq \Psi(1),$$

and is easily reduced to

$$0 \leq t \leq 1 - (1-ay)^s. \qquad (6.51)$$

For the series (6.26) to be usable, $t/(1-ay)^s$ cannot exceed 1. Requiring $-\bar{f}(-z)$ (cf. 6.14) to have a positive second derivative would impose a

limitation of the same order of magnitude. In combination with (6.51) the bound on t requires that

$$1/2 < (1-a)^s. \tag{6.52}$$

For $s = 3$ we know that $a < .2$ will suffice. However, for large s (6.52) implies that $a < 3/s$. Even with this restriction, we will see in the next subsection that the exponential profile fits within the theory, treatable either as a limit as s grows and a decreases, or through a direct computation.

In this section the variable t is most severely restricted by (6.51) when y is near 0 (compare 6.19). While the functions $f_1^+(y,t) = -f_1(y,-t)$ and $h_1^+(y,t) = -h_1(y,-t)$ are well defined in the domain described by (6.20), it will be necessary to restrict the range of t in order to satisfy condition A. The estimates here are simplified somewhat by an alternation in the sign of the terms in the series expansion of f and g. To begin, we show that the terms in the series for f_1^+ alternate in sign and decrease in size for $k > 3$. It will suffice, noting (6.14) and (6.26), to show that for $k > 3$ and $0 < \xi = 1/s < 1/3$,

$$\frac{1}{k!}\,[P_k(\xi) - P_k(2\xi)] \tag{6.53}$$

is non-negative and non-increasing with increasing k. Using (6.22), one reduces the question to showing that

$$0 < \frac{1}{k+1}\,[P_k(\xi)(\xi+k-1) - P_k(2\xi)(2\xi+k-1)] < P_k(\xi) - P_k(2\xi),$$

or that

$$\frac{(\xi-1)\xi(\xi+1)\cdots(\xi+k-2)}{(2\xi-1)2\xi(2\xi+1)\cdots(2\xi+k-2)} < \frac{2-2\xi}{2-\xi} < 1. \tag{6.54}$$

A short computation shows (6.54) to hold for $k = 3$. For larger k the left side of (6.54) becomes smaller, so that the desired result holds for all $k > 3$. It follows that

$$f_1^+ > \frac{s-3}{2s} \frac{t^2}{(1-ay)^{s+1}} + \frac{7-s^2}{s^2} \frac{t^3}{3!(1-ay)^{2s+1}},$$

for $0 < t < 1 - (1-a)^s$. Since $(1-a)^s > .5$, $f_1^+ > \alpha t^2$ holds with

$$\alpha = \frac{s-3}{4s} \tag{6.55}$$

in the domain,

$$\Gamma = \{(y,t) \mid 0 < y < 1,\ 0 < t < \frac{3(s-3)s}{8(s^2-7)}\}. \tag{6.56}$$

Verification of AI in Γ is completed as before. Again we see that the range of t vanishes as s approaches 3.

Since $P_k(\xi)/k!$ decreases with increasing k for $\xi = 2/s < 2/3$, one sees from (6.30) that the series for $h_1^+(y,t) = -h_1(y,-t)$ has alternating signs and decreasing size. Thus

$$\sigma' t^2 < h_1^+(y,t) < \sigma t^2$$

holds in Γ, for $\sigma = 0$ and

$$\sigma' = -\frac{a^2(s-1)(s-2)}{s(1-a)^{s+2}}.$$

The t derivative of h_1^+ is merely (6.32) with t replaced by $-t$ and is always positive for $s > 3$, so AII can be seen to hold.

To satisfy condition AIII we again arbitrarily take $\theta = 3/4$ and note that $t < 2/3$ in Γ from (6.56). Then for $k > 4$, $[(k+1)\theta-2]t^k$ is decreasing with k and we can exploit the alternation in the series for $\lambda f_1 - h_1$ in satisfying (6.34) (compare (6.45), (6.46)). Here $\lambda > \sigma/\alpha = 0$ and the analogue of (6.46) will be satisfied if

$$\lambda\left(\frac{1}{4}\frac{s-3}{2s^2} t^2 - \frac{s^2-7}{s^3} \frac{t^3}{3!(1-a)^s} + \text{postive terms}\right)$$

$$+ \frac{a^2(s-1)(s-2)}{s(1-a)}\left(\frac{3}{4} t^2 - \frac{2(2+s)}{s} \frac{t^3}{3!(1-a)^s} + \text{positive terms}\right) > 0.$$

Since $(1-a)^s > .5$, one easily checks that the last inequality holds in Γ. The discussion surrounding formula (6.40) is now relevant, though with ϕ replaced by $-\phi$ in that formula, corresponding to waves of elevation.

The Exponential Profile

If the density in the undisturbed state is taken to be

$$\rho(y) = \rho_0 e^{-\beta y} \tag{6.57}$$

then the pseudo-stream-function corresponding to (6.3) is

$$\Psi(y) = \frac{2\rho_0^{1/2} c}{\beta} \left(1 - e^{-\frac{1}{2}\beta y}\right). \tag{6.58}$$

A computation similar to that carried out for the density (6.1) shows that in this case the basic equation (6.13) (or (2.1)) has

$$f^+(y,\phi) = e^{-\frac{1}{2}\beta y}\left(1 + e^{\frac{1}{2}\beta y}\phi\right)\log\left(1 + e^{\frac{1}{2}\beta y}\phi\right), \tag{6.59}$$

$$h^+(y,\phi) = \frac{\beta^2}{4}\phi, \tag{6.60}$$

and

$$\lambda = g\beta/c^2, \tag{6.61}$$

with $j = -\gamma = -2\rho_0^{1/2}c/\beta$. One can now proceed to verify condition A as before. However, we have already done this, for (6.57) is obtainable as a limit

$$\rho_0 e^{-\beta y} = \lim_{r\to\infty} \rho_0 \left(1 - \frac{\beta}{r} y\right)^r$$

with $a = \beta/r$ in (6.1). In terms of s, with $a = \beta/(2s-2)$, $(1-ay)^s$ approaches $e^{-\frac{1}{2}\beta y}$ and f_0^+ approaches 1 as $s \to +\infty$. The expression $s[P_k(2/s) - P_k(1/s)]$ in (6.26) approaches the derivative of P_k at 0,

i.e. $(k-2)!$, and the whole series in (6.26) approaches a series for $f(y,t)$ from which one obtains a series for $f_1^+(y,t) = -f_1(y,-t)$. Of course, this series can also be calculated directly using (6.59).

From (6.29) it is apparent that

$$h_0^+(y) = \frac{\beta^2}{4},$$

while from (6.30), $h_1^+ = 0$, since each coefficient is $O(1/s)$, as $s \to +\infty$. Previously we had the restriction $a < 3/s$ for large s so that now it is required that $\beta = 2as < .6$. The restriction on t from (6.56) allows us to take $0 \leq t < 3/8$ and from (6.55), $\alpha = 1/4$ will suffice on that interval. The remaining points of condition A carry over in the limit and we therefore obtain periodic and solitary waves of elevation with

$$\psi(x,y) = \frac{2\rho_0^{1/2}c}{\beta}\left[1 - e^{-\frac{1}{2}\beta y} - \phi(x,y)\right], \qquad (6.63)$$

where

$$c = (g\beta/\lambda)^{1/2}.$$

7. CONCLUSION

The hypotheses AI, AII and AIII introduced in section 3 and the additional restrictions in lemmas 4.1 and 4.10 effectively delineate a class of quiescent states of a heterogeneous fluid confined between two rigid horizontal plane boundaries. Wave motion superimposed upon these quiescent states has been contemplated. Provided that the effects of dissipation, diffusion of the heterogeneity, and compressibility may be ignored, on time scales sufficient for the passage of the waves in question, this class of heterogeneous systems has been demonstrated to support two-dimensional steady-wave motion corresponding generally to surface cnoidal and solitary waves. It deserves emphasis that these waves, which are solutions of the full Euler equations, need not have small amplitude.

In this section we discuss some implications and drawbacks of the present theory, and point to directions for future investigation.

Perhaps the most obvious drawback of our analysis is the absence of a practical set of sufficient conditions that a given system fall within the confines of the class covered by the theory. Such conditions would ideally be expressed directly in terms of the undisturbed density stratification ρ and the undisturbed fluid velocity U. As the examples studied in section 6 amply demonstrate, our theory does have utility in a considerable range of situations. Nevertheless, checking the hypotheses of the theory can be somewhat tedious.

It should also be admitted that variation of density that features a relatively rapid change in the middle of the vertical extent of the flow domain, say joining two homogeneous layers of fluid of constant density ρ_T and ρ_B (with $\rho_T < \rho_B$) situated at the top and bottom boundary, respectively, have not been treated by our theory. Such density variations

are addressed in the complementary paper of Turner (1981). The latter reference even includes results pertaining to the much-studied two-layer problem (a layer of light fluid resting upon a layer of heavier fluid) which has a discontinuous undisturbed density profile. Similarly, Ter-Krikorov (1963) was able to treat systems having an arbitrary smooth undisturbed density profile. Ter-Krikorov's techniques limit his results to small amplitude solitary and cnoidal waves. While the variational techniques used by Turner allow for finite-amplitude waves, the required estimates for the quasilinear elliptic equation arising in his analysis effectively limit the applicability to small amplitudes. We faced a similar situation here in having to limit the ranges of energies or speeds. However, the linear elliptic estimates intervening in section 4 are not such as to vitiate our claim to have found "finite-amplitude waves". In none of the studies under discussion is the entire collection of allowable solutions exhausted. In the case of surface waves on fluid of constant density one has a more satisfactory global picture. Amick and Toland (1980) have shown that there is a connected set of solutions joining a "wave of greatest height" to a trivial flow.

Another possible objection to the present theory is the fixed upper boundary that intrudes in our formulation. This simplification has a long history (cf. Keulegan 1953, Long 1956, Benjamin 1966, 1971). Despite these precedents, a fixed upper boundary certainly seems somewhat artificial when one thinks of waves in the ocean or atmosphere. Peters and Stoker (1960) obtained a formalism for internal solitary waves which allowed a free upper surface. Benjamin (1966, §4) also gives theoretical results, some of which relate to the situation where the upper surface is left free. He notes that the resultant change in boundary conditions that is implied in passing from a fixed to a free upper surface can fundamentally alter the character of the

flow. Especially considering this latter point, a rigorous theory that takes account of a free upper surface, or which is set in a semi-infinite domain where the upper boundary has receded to infinity, would seem a worthwhile undertaking.

Another aspect of internal-wave theory that deserves attention is the relationship of solutions of the Euler equations to solutions of the various model equations that formally apply to small-amplitude, long-wavelength disturbances. Because of the extra length scale, the vertical width of the stratified layer, a number of essentially different models arise, a situation at variance with the usual theory of small-amplitude long suface waves. Depending on the relationship between the varius parameters that intervene in the regime at hand, we may obtain the Korteweg-de Vries equation (Benney, 1966), Benjamin's (1967) equation, or the equation derived by Joseph (1977). (The latter equation reduces to the former two in certain limiting cases.) It is not intended to initiate a detailed discussion of these equations. However, it is interesting to recall that the solitary-wave solutions of these models play a distinguished role in the evolution of general classes of initial data, in that the initial data resolves itself into a sequence of solitary waves followed by a decaying dispersive tail (cf. Miura 1976). There are not results available now providing rigorous comparisons of solutions of these model equations with solutions of the Euler equations. Notwithstanding, such model equations have been observed to give quite acceptable predictions (cf. Bona, Pritchard and Scott 1981, and the references contained therein), so it is not unreasonable to conjecture that solitary internal waves do play a somewhat special role in the long-term evolution of certain classes of disturbances. (Some additional evidence in favour of this conjecture will be introduced presently.) If valid, this proposition certainly heightens interest in solitary-wave solutions of the Euler equations.

The waves whose existence is demonstrated in section 5 deserve, at least provisionally, the appellation "solitary wave". They accord with Scott Russell's (1845) original conception, a single-crested wave which is sensibly localized in space and which propagates at a constant velocity and without change of form. The surface water waves regarded in the field and in the laboratory by Scott Russell are remarkably stable. To fully justify the designation "solitary wave" for the waves dealt with herein, a satisfactory theory of stability pertaining to these waves must be forthcoming. Alas, at this juncture even a uniqueness result may only be surmised. Therefore, this issue is left aside in the present work, though its resolution would certainly be of considerable interest. The solitary-wave solutions of the Korteweg-de Vries equation and of Benjamin's equation are known to be stable (cf. Benjamin 1972, Bona 1975 and Bennett et. al. 1981). Insofar as these model evolution equations reflect the Euler equations, the last-mentioned results may be construed as evidence in favour of the stability of the waveforms dealt with in section 5.

Experimental and field observations point even more convincingly to the stability of the internal waves addressed earlier. We point to the laboratory studies of Davis and Acrivos (1967), Walker (1972) and Maxworthy (1979, 1980). On a grander scale, the recent observations of internal waves off the coasts of the continental U.S.A. (cf. Apel et. al. 1975), in the Andaman sea (Apel 1979, Osborn and Burch 1980), and elsewhere, present themselves. Internal waves with peak to trough amplitudes as large as 100m are represented in these rather spectacular reports from the field. Some of these waves have been interpreted as internal solitary waves, and indeed the evidence for this presupposition is substantial, if not compelling. From the view of the present discussion, these observations tend especially to reinforce two

points. The first is the exceptionally stable nature of internal solitary waves. The second is the distinguished role played by the solitary wave in the long-term evolution of certain types of disturbances, as would be predicted by the various model equations. As the waves in question probably obey the Euler equations more closely than the simplified model equations, there is an inference that the Euler equations may exhibit the forementioned phenomenon of resolving perturbations into a sequence of solitary waves*.

Finally, it deserves emphasising that the periodic, or cnoidal waves whose existence has been assured in theorems 3.2 and 3.4 do not satisfy the boundary conditions under which Yih's equation (2.5) was derived. In consequence, while they may be used to define solutions of the Euler equations, no direct physical significance can be imputed to these solutions. Nonetheless, it has been argued by Benjamin (1966, §3) that such periodic wavetrains may be generated in the lee of an obstacle introduced into a primary flow, or on the back of an internal undular bore. In both these cases, a change in energy is needed in the generation region, near the obstacle or near the front of the bore, respectively, in order that such waves may arise. In consequence, this sort of situation is beyond the scope of the present study, both in regard to predicting which wavetrain would be manifested in a given configuration, and in regard to the possibility of an exact analyis. In any case, the periodic wavetrains present some interest, if for no other reason than their approach to the solitary wave, as their period grows indefinitely large.

---

* We are not suggesting that the Euler equations are completely integrable. The recent work of Benjamin and Olver (1981) makes this an unlikely possibility. However, it appears that wave equations may have the property of resolving initial data into solitary waves without being completely integrable (cf. Bona 1981).

Appendix

In this section we set forth some results, used in section 3, regarding symmetrization (or rearrangement) of functions.

Let T be a triangulation of the rectangle $\bar{\Omega}_k = \{(x,y) \mid -k \leq x \leq k, 0 \leq y \leq 1\}$, i.e., a collection of closed triangles contained in $\bar{\Omega}_k$, having union equal to $\bar{\Omega}_k$ and having the property that any pair intersect in a common edge or vertex, if at all. Let u be a function defined on $\bar{\Omega}_k$. We say u is <u>piecewise linear</u> (PL) if it is continuous in $\bar{\Omega}_k$ and if there is a triangulation T of $\bar{\Omega}_k$, as described above, such that u is linear in x and y on each triangle of T. Starting from the fact that $C^1(\Omega_k)$ is dense in $W^{1,2}(\Omega_k)$, and approximating a C^1 function by a PL function in the $W^{1,2}$ norm, one shows that PL functions are dense in $W^{1,2}(\Omega_k)$. We denote by PL^+ the collection of $u \in PL$ for which $u_x \neq 0$ a.e. Suppose $s \in PL$ has $s_x \neq 0$ a.e. By considering $u(x,y) + \varepsilon s(x,y)$, for suitable small ε, one sees that PL^+ is dense in $W^{1,2}(\bar{\Omega}_k)$. All we require for our variational principles is a dense class. At the same time, the symmetrization which we next define is very easily analyzed in the class PL^+. One can extend the results on symmetrization to all of $W^{1,2}$ as was done by Fraenkel and Berger (1974). However, the weak limits involved can equally well be postponed until working with a variational principle as we do in this paper.

Suppose u is in PL^+ on Ω_k. For each fixed y with $0 \leq y \leq 1$, let

$$\mu(a,y) = m\{x \mid u(x,y) > a\},$$

where m is 1-dimensional Lebesgue measure and a lies between the minimum and maximum of u. The measure in question is merely the sum of the lengths of a finite set of intervals and since $u_x \neq 0$ a.e., μ will be continous in a and y. For each y the value of μ strictly decreases as a

increases. For $a = \max u$, $\mu = 0$, while if $a = \min u$, $\mu = 2k$. We seek a function $\tilde{u}$ which is even in x, decreases in x for $0 \leq x \leq k$, and satisfies

$$m\{x \mid \tilde{u}(x,y) > a\} = \mu(a,y).$$

If, for a given y, $a(\mu,y)$ denotes the function inverse to $\frac{1}{2}\mu(a,y)$, then the even function $\tilde{u}(x,y)$ defined by

$$\tilde{u}(x,y) = a(x,y),$$

for $0 \leq x \leq k$, clearly meets our criteria. Note that if $v = v(x,y)$ is another PL function, then $u \leq v$ implies $\tilde{u} \leq \tilde{v}$.

<u>Lemma</u> 1. Let u and w be in PL^+. Then

$$\|\tilde{u}-\tilde{w}\|_{L^\infty(\Omega_k)} \leq \|u-w\|_{L^\infty(\Omega_k)}.$$

<u>Proof</u>. Let $\|u-w\|_{L^\infty} = \delta$. Then $w \leq u + \delta$ so $\tilde{w} \leq \widetilde{u+\delta}$. If $\tilde{u}(x) = a$ then $2x = m\{u > a\} = m\{u + \delta > a + \delta\}$, so $(\widetilde{u+\delta})(x) = a+\delta = \tilde{u}(x) + \delta$. Thus $\tilde{w} \leq \tilde{u}+\delta$. Similarly, $\tilde{u} \leq \tilde{w} + \delta$, so $\|\tilde{u}-\tilde{w}\|_{L^\infty} \leq \delta$.

For $u \in PL^+$ the absolute value $|u|$ is in PL^+. Define (compare definition 2.1)

$$v = |u|$$

and

$$\hat{u} = \tilde{v}.$$

We call $\hat{u}$ the <u>symmetrization</u> of u, after Steiner (cf. Polya and Szego 1951, note A). It follows immediately from lemma 1 that

$$\|\hat{u}-\hat{z}\|_{L^\infty(\Omega_k)} \leq \|u-z\|_{L^\infty(\Omega_k)} \tag{Ap. 1}$$

for PL^+ functions u and z defined on Ω_k.

Our use of symmetrization hinges on the behavior of certain integrals of symmetrized functions. First, suppose $G(y,u)$ is continuous in y and u, for $0 \leq y \leq 1$ and any real u. For $u \in PL^+$ an elementary argument shows that

$$\int_{\Omega_k} G(y,u(x,y))dxdy = -\int_0^1 \int_{u_{min}}^{u_{max}} G(y,a)d\mu(a,y)dy,$$

and hence that

$$\int_{\Omega_k} G(y,u)dxdy = \int_{\Omega_k} G(y,\tilde{u})dxdy$$

(cf. Polya and Szego 1951, p. 185). Since the function $\hat{u}$ has the same measure distribution μ as $|u|$, it follows that

$$\int_{\Omega_k} G(y,|u|)dxdy = \int_{\Omega_k} G(y,\hat{u})dxdy.$$

If $G(y,|u|) = G(y,u)$, i.e. if G is even in u, then

$$\int_{\Omega_k} G(y,u)dxdy = \int_{\Omega_k} G(y,\hat{u})dxdy. \qquad \text{(Ap. 2)}$$

Another relation we need for a function u in PL^+, which is $2k$ periodice in x, is the inequality

$$\int_{\Omega_k} |\nabla u|^2 dxdy \geq \int_{\Omega_k} |\nabla\hat{u}|^2 dxdy, \qquad \text{(Ap. 3)}$$

which follows from (Polya and Szego 1951, p. 186) since $|\nabla u|^2 = |\nabla|u||^2$ a.e. It should be noted that (Ap. 3) will not hold for an arbitrary PL^+ function on Ω_k. For the cited proof to apply it must be that each line y = constant, u = constant which intersects the graph of u, intersects it in at least two points. A function which is $2k$ periodic in x has this property on $\bar{\Omega}_k$, and since we consider only such functions, (Ap. 3) is certainly at our disposal throughout.

REFERENCES

1. Adams, R. A. 1975 Sobolev Spaces. Academic Press: New York.

2. Agmon, S., Douglis, A. and Nirenberg, L. 1959 Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I. Comm. Pure Appl. Math. 12, 623.

3. Ambrosetti, A. and Rabinowitz, P. H. 1973 Dual variational methods in critical point theory and applications. J. Funct. Anal. 14, 349.

4. Amick, C. J. and Toland, J. F. 1981 On solitary water-waves of finite amplitude. Arch. Rat. Mech. Anal. 76, 9.

5. Apel, J. R. 1979 Observations of internal wave surface signatures in ASTP photographs. In Appolo-Soyez Test Project, NASA Summary Sci. Report no. SP-412.

6. Apel, J. R., Byrne, H. M., Proni, J. R. and Charnell, R. L. 1975 Observations of oceanic internal and surface waves from the earth resources technology satellite. J. Geophys. Res. 80, 865.

7. Benjamin, T. B. 1966 Internal waves of finite amplitude and permanent form. J. Fluid Mech. 25, 241.

8. Benjamin, T. B. 1967 Internal waves of permanent form in fluids of great depth. J. Fluid Mech. 29, 559.

9. Benjamin, T. B. 1971 A unified theory of conjugate flows. Phil. Trans. R. Soc., London A 269, 587.

10. Benjamin, T. B. 1972 The stability of solitary waves. Proc. R. Soc. Lond. A 328, 153.

11. Benjamin, T. B. 1973 An exact theory of finite steady waves in continuously stratified fluids. Fluid Mechanics Research Institute, University of Essex, report no. 48.

12. Benjamin, T. B. and Olver, P. J. 1981 Hamiltonian structures, symmetries and conservation laws for water waves. University of Wisconsin-Madison, Math. Res. Ctr. TSR #2266, 1981.

13. Bennett, D. D., Bona, J. L., Brown, R. W., Stansfield, S. E. and Stronghair, J. D. 1981 The stability of internal solitary waves in stratified fluids. (to appear).

14. Benney, D. J. 1966 Long non-linear waves in fluid flows. J. Math. Phys. **45**, 52.

15. Bona, J. L. 1975 On the stability theory of solitary waves. Proc. R. Soc., Lond. A **344**, 363.

16. Bona, J. L. 1981 On solitary waves and their role in the evolution of long waves. In Applications of nonlinear analysis in the physical sciences (ed. H. Amann, N. Bazley and K. Kirchgässner) London: Pitman.

17. Bona, J. L., Pritchard, W. G. and Scott, L. R. 1980, Solitary-wave interaction. Phys. Fluids **23**, no. 3, 438.

18. Bona, J. L., Pritchard, W. G. and Scott, L. R. 1981 An evaluation of a model equation for water waves. Phil. Trans. R. Soc. Lond. A **302**, 457.

19. Crandall, M. G. and Rabinowitz, P. H. 1970 Nonlinear Sturm-Liouville eigenvalue problems and topological degree. J. Math. Mech. **19**, 1083.

20. Davis, R. E. and Acrivos, A. 1967 Solitary internal waves in deep water. J. Fluid Mech. **29**, 593.

21. Dubreil-Jacotin, M. L. 1937 Sur les théorèmes d'existence relatifs aux ondes permanentes périodiques a deux dimensions dans les liquides hétérogènes. J. Math. Pures Appl. (9) **19**, 43.

22. Fraenkel, L. E. and Berger, M. S. 1974 A global theory of steady vortex rings in an ideal fluid. Acta Mathematica **132**, 13.

23. Friedrichs, K. O. and Hyers, D. H. 1954 The existence of solitary waves. Communs. Pure Appl. Math. **7**, 517.

24. Friedrichs, K. O. 1934 Über ein minimumproblem fur potentialströmungen mit freiem rande. Math. Ann. 109, 60.

25. Garabedian, P. R. 1965 Surface waves of finite depth. Journal d'Anal. Math. 14, 161.

26. Gilbarg, D. and Trudinger, N. S. 1977 Elliptic partial differential equations of second order. Berlin: Springer-Verlag.

27. Howard, L. N. 1961 Note on a paper of John W. Miles. J. Fluid Mech. 10, 509.

28. Imberger, J., Thompson, R. and Fandry, C. 1976 Selective withdrawal from a finite rectangular tank. J. Fluid Mech. 78, 489.

29. Ince, E. L. 1927 Ordinary differential equations. London: Longmans, Green & Co.

30. Joseph, R. I. 1977 Solitary waves in a finite depth fluid. J. Phys. A 10, no. 12, L225.

31. Keulegan, G. H. 1953 Characteristics of internal solitary waves. J. Nat. Bur. Stand. 51, 133.

32. Korteweg, D. J. and de Vries, G. 1895 On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary wave. Phil Mag. 39, 422.

33. Kotchin, N. 1928 Determination rigoureuse des ondes permanentes d'ampleur finie àla surface de séparation de deux liquides de profondeur finie. Math. Ann. Volume #582.

34. Krasnoselskii, M. A. 1964 Topological methods in the theory of nonlinear integral equations, MacMillan. New York: MacMillan.

35. Lavrentief. M. A. 1943 On the theory of long waves; 1947 A contribution to the theory of long waves. In Amer. Math. Soc. Translation No. 102, (1954), Providence, RI.

36. Long, R. R. 1953 Some aspects of the flow of stratified fluids. Part I. A theoretical investigation. Tellus 5, 42.

37. Long, R. R. 1956 Solitary waves in the one- and two-fluid systems. Tellus 8, 460.

38. Mahony, J. J. and Pritchard, W. G. 1981 Withdrawal from a reservoir of stratified fluid. Proc. R. Soc. Lond. A 376, 499.

39. Maxworthy, T. 1979 A note on the internal solitary waves produced by tidal flow over a three-dimensional ridge. J. Geophys. Res. 84, 338.

40. Maxworthy, T. 1980 On the formation of nonlinear internal waves from the gravitational collapse of mixed regions in two and three dimensions. J. Fluid Mech. 96, 47.

41. Miles, J. W. 1961 On the stability of heterogeneous shear flows. J. Fluid Mech. 10, 496.

42. Miles, J. W. 1963 On the stability of heterogeneous shear flows. Part 2. J. Fluid Mech. 16, 209.

43. Miura, R. M. 1976 The Korteweg-de Vries equation: a survey of results. SIAM Rev. 18, 412.

44. Osborne, A. R. and Burch, J. L. 1980 Internal waves in the Andaman Sea. Science 208, 451.

45. Peters, A. S. and Stoker, J. J. 1960 Solitary waves in liquids having non-constant density. Communs. Pure Appl. Math. 8, 115.

46. Polya, G. and Szego, G. 1951 Isoperimetric inequalities in mathematical physics. Annals of Math. Studies 27: Princeton University Press.

47. Scott, A. C., Chu, F. Y. F. and McLaughlin, D. W. 1973 The soliton: a new concept in applied science. Proc. I.E.E.E. 61, no. 10, 1443.

48. Russell, J. S. 1844 Report on waves. Rep. 14th Meeting of the British Association for the Advancement of Science, p. 311. (London: John Murray 1845).

49. Shen, M. C. 1964 Solitary waves in compressible media. New York Univ. Inst. Math. Sci., Rep. IMM-NYU 325.

50. Shen, M. C. 1965 Solitary waves in running gases. New York Univ. Inst. Math. Sci., Rep. IMM-NYU 341.

51. Stakgold, I. 1968 Boundary value problems of mathematical physics, vol. II. New York: Macmillan.

52. Ter-Krikorov, A. M. 1963 Théorie exacte des ondes longues stationnaires dans un liquide hétérogene. J. Mecanique 2, 351.

53. Turner, R. E. L. 1981 Internal waves in fluids with rapidly varying density. Annali della Scuola Normale - Pisa. Ser. IV, 8, 513.

54. Vainberg, M. M. 1964 Variational methods for the study of nonlinear operators. San Francisco: Holden-Day.

55. Walker, L. R. 1973 Interfacial solitary waves in a two-fluid medium. Phys. of Fluids. 16, no. 11, 1796.

56. Yih, C.-S. 1958 On the flow of a stratified fluid. Proc. U.S. Nat. Congr. Appl. Mech. 3rd, 857.

57. Yih, C.-S. 1960 Exact solutions for steady two-dimensional flow of a stratified fluid. J. Fluid Mech. 9, 161.

58. Zeidler, E. 1971 Beiträge zur theorie und praxis freier randwertaufgaben. Berlin: Akademie-Verlag.

JLB/DKB/RELT/jvs

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: #2401	2. GOVT ACCESSION NO.: AD A120 760
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): Finite-Amplitude Steady Waves in Stratified Fluids	5. TYPE OF REPORT & PERIOD COVERED: Summary Report - no specific reporting period
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): J. L. Bona, D. K. Bose, R. E. L. Turner	8. CONTRACT OR GRANT NUMBER(s): MCS-8002327 & MCS-7904426; DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: Work Unit Number 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS: (see Item 18 below)	12. REPORT DATE: July 1982
	13. NUMBER OF PAGES: 80
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

U. S. Army Research Office
P. O. Box 12211
Research Triangle Park
North Carolina 27709

and

National Science Foundation
Washington, DC 20550

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Internal wave, solitary wave, cnoidal wave, critical point, symmetrization, bifurcation

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

An exact theory regarding solitary internal gravity waves in stratified fluids is presented. Two-dimensional, inviscid, incompressible flows confined between plane horizontal rigid boundaries are considered. Variational techniques are used to demonstrate that the Euler equations possess solutions that represent progressing waves of permanent form. These are analogous to the surface, solitary waves so easily generated in a flume. Periodic wave trains of permanent form, the analogue of the classical cnoidal waves, are also found. Moreover, internal solitary-wave solutions are shown to arise as the limit of cnoidal wave trains as the period length grows unboundedly.

END

FILMED

1-83

DTIC